# Final Review Session Problems

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**Exercise 1.** The integral form of the Schrödinger equation reads

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int g(\mathbf{r} - \mathbf{r}')V(\mathbf{r}')V(\mathbf{r}')\psi(\mathbf{r}')\, \mathrm{d}^3\mathbf{r}',$$

where

$$g(\mathbf{r}) = -\frac{m}{2\pi\hbar^2} \cdot \frac{e^{ikr}}{r}$$

is the Green's function for the Schrödinger equation.

a) Use the method of successive approximations to write $\psi(\mathbf{r})$ as a series in the incident wavefunction $\psi_0(\mathbf{r})$.

b) Truncate the Born series you obtain after the second term to get the first Born approximation. Assuming the potential is localized near $\mathbf{r}' = 0$, we can write

$$\frac{e^{ik(\mathbf{r}-\mathbf{r}')}}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{e^{ik\mathbf{r}}}{r} e^{-i\mathbf{k}\cdot\mathbf{r}'}.$$

Using this and the definition of $f(\theta)$,

$$\psi(\mathbf{r}) = Ae^{ikz} + f(\theta)\frac{e^{ikr}}{r},$$

determine $f(\theta)$.

c) In Griffiths, we find that for a potential $V(r) = V_0/r$, $f_{\text{point}}(\theta) = -\frac{2mV_0}{\hbar^2 q^2}$, where $\mathbf{q} = \mathbf{k}' - \mathbf{k}$. If $V(\mathbf{r}) = -e^2 Z/r$ for an electron scattering off a point charge of charge $Ze$, how would $f(\theta)$ change if instead the electron scatters off a spherical nucleus of radius $a$, charge $Ze$, and uniform charge density? Your answer should be of the form

$$f(\theta) = f_{\text{point}}(\theta) \cdot F(q),$$

where $F(q)$ is the **form factor** of the nucleus.

d) If you haven't done so already, calculate $F(q)$ explicitly.

e) From scattering high-energy electrons at nuclei, the actual form factor is measured to be

$$F(q) = \frac{Ze}{(1 + q^2 {a_N}^2)^2},$$

where $a_N \approx 0.26$ fm. If the inverse Fourier transform of $\frac{1}{(1+x^2)^2}$ is $e^{-|x|}$, what does that tell you about the size and charge density of the proton?

**Exercise 2.** Consider a 1D harmonic oscillator of angular frequency $\omega_0$ that is perturbed by a time-dependent potential $V(t) = bx\cos(\omega t)$, where $x$ is the displacement of the oscillator from equilibrium. Evaluate $\langle x \rangle$ by time-dependent perturbation theory. Discuss the validity of the result for $\omega \approx \omega_0$ and $\omega$ far from $\omega_0$.

**Exercise 3.** *Griffiths 11.33* The spontaneous emission of the 21-cm hyperfine line in hydrogen is a magnetic dipole transition with rate

$$\Gamma = \frac{\omega^3}{3\pi\varepsilon_0 \hbar c^3} \left| \left\langle B \left| \frac{\vec{\mu}_e + \vec{\mu}_p}{c} \right| A \right\rangle \right|^2,$$

where

$$\begin{aligned} \vec{\mu}_e &= -\frac{e}{m_e}\vec{S}_e \\ \vec{\mu}_p &= \frac{5.59e}{2m_p}\vec{S}_p. \end{aligned}$$

On midterm 1, you showed the triplet has slightly higher energy than the singlet. Calculate (approximately) the lifetime of this transition.

**Exercise 4.** Consider a dynamical variable $\xi$ that can take only two values, 1 or -1 (for example, $\sigma_z$ is such an operator for a spin 1/2 particle). Denote the corresponding eigenvectors as $|+\rangle$ and $|-\rangle$. Now, consider the following states.

a) The one-parameter family of pure states

$$|\theta\rangle = \sqrt{\frac{1}{2}}(|+\rangle + e^{i\theta}|-\rangle)$$

for any real $\theta$.

b) The nonpure state

$$\rho = \frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|).$$

Show that $\langle \xi \rangle = 0$ in all of these states. What, if any, are the physical differences between these various states, and how could they be measured?

**Exercise 5.** In the homework, you showed that the most general density matrix for a spin 1/2 particle is $\rho = \frac{1}{2}(1 + \vec{a} \cdot \vec{\sigma})$, where $\vec{a}$ is some 3-vector. If the system has a magnetic moment $\vec{\mu} = \frac{1}{2}\gamma\hbar\vec{\sigma}$ and is in a constant magnetic field $\vec{B}$, calculate $\rho(t)$. Describe the result geometrically in terms of the variation of the vector $\vec{a}$.