Week 11 Worksheet Solutions
The Einstein Equation and Linearized Gravity

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Exercise 1. Linearized Gravity. In this exercise, you will carefully work through the derivation of linearized gravity. Recall that in this regime, we suppose the metric has the form

$$g_{ij} = \eta_{ij} + \gamma_{ij},$$

where η_{ij} is the Minkowski metric and γ_{ij} is a small perturbation. Linearized gravity means we ignore all contributions to relevant quantities that are of order γ^2. By convention, raising and lowering of indices is done with η instead of with g.

a) Check that the inverse metric is

$$g^{ij} = \eta^{ij} - \gamma^{ij}.$$

b) The Christoffel symbols and Ricci tensor are

$$\begin{aligned} \Gamma^k_{ij} =& \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) \\ R_{ij} =& \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k{}_{kj} + \Gamma^k_{ij} \Gamma^l{}_{kl} - \Gamma^k_{lj} \Gamma^l_{ki}, \end{aligned}$$

where e.g. $\Gamma^k{}_{jk} = \sum_k \Gamma^k_{jk}$ is the contraction. Compute the Christoffel symbols in linearized gravity, and show that the Ricci tensor to first order in γ is

$$R^{(1)}_{ij} = \frac{1}{2} \left(\partial^l \partial_i \gamma_{jl} + \partial^l \partial_j \gamma_{il} - \partial^2 \gamma_{ij} - \partial_i \partial_j \gamma \right),$$

where $\gamma = \gamma^k{}_k$ is the trace of γ and $\partial^2 = \partial^i \partial_i$ is the d'Alembertian.
Hint: Argue immediately that the Γ^2 terms in R_{ij} are 0, without doing any computations with them!

c) The Einstein tensor to first order in γ is

$$G^{(1)}_{ij} = R^{(1)}_{ij} - \frac{1}{2} \eta_{ij} R^{(1)},$$

where $R = R^i{}_i$ is the Ricci scalar. Define

$$\bar{\gamma}_{ij} = \gamma_{ij} - \frac{1}{2}\eta_{ij}\gamma,$$

and substitute this into the Einstein tensor to find

$$G^{(1)}_{ij} = \frac{1}{2}\left(\partial^l \partial_j \bar{\gamma}_{il} + \partial^l \partial_i \bar{\gamma}_{jl}\right) - \frac{1}{2}\partial^2 \bar{\gamma}_{ij} - \frac{1}{2}\eta_{ij}\partial^l \partial^k \bar{\gamma}_{kl}.$$

Hints: Show first that

$$\partial^l \partial_j \gamma_{il} + \partial^l \partial_i \gamma_{jl} = \partial^l \partial_j \bar{\gamma}_{il} + \partial^l \partial_i \bar{\gamma}_{jl} + \partial_i \partial_j \gamma.$$

Next, show that

$$\eta_{ij}\partial^l \partial^k \gamma_{kl} = \eta_{ij}\partial^l \partial^k \bar{\gamma}_{kl} + \frac{1}{2}\partial^2 \gamma.$$

d) General relativity has gauge transformations, similar to gauge transformations in electromagnetism. Recall that the 4-potential A has a gauge freedom

$$A_\mu \to A_\mu + \partial_\mu \lambda$$

defined by scalar fields λ. Linearized gravity has a gauge freedom

$$\gamma_{ij} \to \gamma_{ij} + \partial_i v_j + \partial_j v_i$$

defined by vector fields $v = v^i \partial_i$. Show that under a gauge transformation defined by v

$$\partial^j \bar{\gamma}_{ij} \to \partial^j \bar{\gamma}_{ij} + \partial^2 v_i.$$

Thus, by solving

$$\partial^2 v_i = -\partial^j \bar{\gamma}_{ij}$$

for v_i, we are free to set

$$\partial^j \bar{\gamma}_{ij} = 0.$$

This is called **Lorenz gauge**.

e) Show that in the Lorenz gauge, the linearized Einstein equation

$$G^{(1)}_{ij} = 8\pi T_{ij}$$

becomes

$$\partial^2 \bar{\gamma}_{ij} = -16\pi T_{ij}.$$

a) The cross terms cancel, $\eta^{ij}\eta_{jk} = \delta^i_k$, and we can ignore the term quadratic in γ.

b) We can ignore the terms with two factors of Γ since Γ is linear in γ; hence, Γ^2 is quadratic in γ and can be ignored. Thus, we compute

$$R^{(1)}_{ij} = \frac{1}{2}\eta^{kl}\left(\partial_k\partial_i\gamma_{jl} + \partial_k\partial_j\gamma_{il} - \partial_k\partial_l\gamma_{ij} - \partial_i\partial_k\gamma_{jl} - \partial_i\partial_j\gamma_{kl} + \partial_i\partial_l\gamma_{kj}\right).$$

Now, notice that the first and fourth terms cancel, which gives the result after raising the appropriate indices with η^{kl}.

c) Again, we compute

$$G^{(1)}_{ij} = \frac{1}{2}\left(\partial^l\partial_j\gamma_{il} - \partial^2\gamma_{ij} - \partial_i\partial_j\gamma + \partial_i\partial^l\gamma_{lj}\right) - \frac{1}{4}\eta_{ij}\left(2\partial^l\partial^k\gamma_{kl} - 2\partial^2\gamma\right).$$

We check the first formula in the hint by a straightforward calculation, noticing that $\eta_{il}\partial^l\partial_j\gamma = \eta_{jl}\partial^l\partial_i\gamma = \partial_i\partial_j\gamma$. Similarly, we check the second formula in the hint by noting that

$$\eta_{ij}\partial^l\partial^k\gamma_{kl} = \eta_{ij}\partial^l\partial^k\bar{\gamma}_{kl} + \frac{1}{2}\eta_{ij}\eta_{kl}\partial^l\partial^k\gamma = \eta_{ij}\partial^l\partial^k\bar{\gamma}_{kl} + \frac{1}{2}\eta_{ij}\partial^2\gamma.$$

Thus, the first, third, and fourth terms of $G^{(1)}_{ij}$ combine by the first formula in the hint. By the second formula we just derived, the second and last terms combine if we add in the extra factor of $\partial^2\gamma$ just derived. This gives the result.

d) We compute

$$\partial^j\bar{\gamma}_{ij} = \partial^j\gamma_{ij} - \frac{1}{2}\partial_i\gamma \to \partial^j\bar{\gamma}_{ij} + \partial^j\partial_i v_j + \partial^2 v_i - \frac{1}{2}\partial_i\partial_j v^j - \frac{1}{2}\partial_i\partial_j v^j.$$

The last two terms are the same, and they exactly cancel the second term. This gives the result.

e) The only term in $G^{(1)}_{ij}$ that isn't of the form $\partial^j\bar{\gamma}_{ij}$ is the $\partial^2\bar{\gamma}_{ij}$ term.

Exercise 2. The Newtonian Limit. Assume that $T = \rho\partial_t \otimes \partial_t$, i.e.

$$T_{\mu\nu} = \rho(\partial_t)_\mu(\partial_t)_\nu,$$

where ∂_t is the vector field in the time direction of our coordinate system. Assume this coordinate system to be global. Further assume that time derivatives of $\bar{\gamma}_{\mu\nu}$ are negligible because *the sources are slowly varying*. In this problem, Greek indices go from $0, \dots, 3$ while Latin indices go from $1, \dots, 3$.

a) Show that the result of Exercise 1(e) becomes

$$\nabla^2\bar{\gamma}_{ij} = 0,$$

where now $i, j \in \{1, 2, 3\}$ and ∇^2 is the usual laplacian on 3-space, while

$$\nabla^2\bar{\gamma}_{00} = -16\pi\rho.$$

Hint: What are the components $(\partial_t)_\mu$ in a coordinate system? Argue that the form for T implies $T_{\mu\nu} = 0$ unless $\mu = 0$ and $\nu = 0$.

b) Argue that the unique solution of $\nabla^2\bar{\gamma}_{ij} = 0$ is $\bar{\gamma}_{ij} = 0$.
Hints: Recall the form of the solution of Laplace's equation in spherical coordinates, and notice that it needs to be well-defined at both $r = 0$ and at $r = \infty$. If $\bar{\gamma}_{ij}$ must be a constant, then we can in fact set it to 0 by a gauge transformation.

c) Denote $(\partial_t)_\mu = t_\mu$, and show that our solution for the perturbed metric $\gamma_{\mu\nu}$ is

$$\gamma_{\mu\nu} = -(4t_\mu t_\nu + 2\eta_{\mu\nu})\varphi,$$

where $\varphi = -\frac{1}{4}\bar{\gamma}_{00}$ satisfies Poisson's equation

$$\nabla^2\varphi = 4\pi\rho.$$

d) The geodesic equation reads

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0.$$

Assume that the 4-velocity of our particle $dx^\mu/d\tau = (1, 0, 0, 0)$, since our particle is moving much slower than the speed of light in the newtonian limit. Thus, show that

$$\frac{d^2x^\mu}{dt^2} = -\Gamma^\mu_{00} = \partial_\mu\varphi.$$

where we approximate $\tau = t$ and ignore time derivatives of φ. Note that this is exactly the classical equation of motion for a particle in a gravitational potential φ,

$$\mathbf{a} = -\nabla\varphi.$$

a) Since $\partial_t = (1, 0, 0, 0)$ and $\partial_0\gamma_{\mu\nu} = 0$, we have that the only nonvanishing component of T is the T_{00} component and we can ignore the time derivatives in the d'Alembertian, i.e. $\partial^2 = \nabla^2$. This gives the desired results.

b) Solutions to Laplace's equation in spherical coordinates are of the form

$$\sum_{l=0}^{\infty}\left(\frac{a^l}{r^{l+1}} + b^l r^l\right)P_l(\cos\theta).$$

Since $\frac{1}{r^l}$ blows up at the origin and r^l blows up at ∞, we find that the only possible solution is a constant. But we can set the constant to 0 by using a gauge transformation.

c) We have that $\bar{\gamma}_{\mu j} = 0$, while $\bar{\gamma}_{00}$ is the only nonzero component. Since

$$\begin{aligned}\gamma_{\mu\nu} &= \bar{\gamma}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{\gamma} \\ &= \bar{\gamma}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{\gamma}_{00} \\ &= \bar{\gamma}_{\mu\nu} - 2\eta_{\mu\nu}\varphi.\end{aligned}$$

Thus, plugging this in to the form for $\gamma_{\mu\nu}$ given in the statement of the problem, we find

$$\bar{\gamma}_{\mu\nu} = -4t_\mu t_\nu\varphi.$$

Indeed, since $t_\mu t_j = 0$, and $t_0 t_0 = 1$, we have that this satisfies the 16 relevant differential equations.

d) The assumption on the 4-velocity implies that

$$\frac{d^2x^\mu}{dt^2} = -\Gamma^\mu_{00},$$

so we just need to compute this component of the Christoffel symbol. By definition, it is

$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\nu}\partial_\nu\gamma_{00},$$

since the other terms are all of the form $\partial_0\varphi$. We can ignore the $\nu = 0$ part since that is also a time derivative. The $\nu = i$ part is

$$-\frac{1}{2}\eta^{\mu i}\partial_i\gamma_{00} = \delta^{\mu i}\partial_i(2-1)\varphi = \delta^{\mu i}\partial_i\varphi = \partial^\mu\varphi,$$

since $\partial^0\varphi = 0$ anyway.