Week 11 Worksheet
The Einstein Equation and Linearized Gravity

Jacob Erlikhman

April 3, 2025

Exercise 1. Linearized Gravity. In this exercise, you will carefully work through the derivation of linearized gravity. Recall that in this regime, we suppose the metric has the form

$$g_{ij} = \eta_{ij} + \gamma_{ij},$$

where η_{ij} is the Minkowski metric and γ_{ij} is a small perturbation. Linearized gravity means we ignore all contributions to relevant quantities that are of order γ^2. By convention, raising and lowering of indices is done with η instead of with g.

a) Check that the inverse metric is

$$g^{ij} = \eta^{ij} - \gamma^{ij}.$$

b) The Christoffel symbols and Ricci tensor are

$$\begin{aligned} \Gamma^k_{ij} =& \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) \\ R_{ij} =& \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k{}_{kj} + \Gamma^k_{ij} \Gamma^l{}_{kl} - \Gamma^k_{lj} \Gamma^l_{ki}, \end{aligned}$$

where e.g. $\Gamma^k{}_{jk} = \sum_k \Gamma^k_{jk}$ is the contraction. Compute the Christoffel symbols in linearized gravity, and show that the Ricci tensor to first order in γ is

$$R^{(1)}_{ij} = \frac{1}{2} \left(\partial^l \partial_i \gamma_{jl} + \partial^l \partial_j \gamma_{il} - \partial^2 \gamma_{ij} - \partial_i \partial_j \gamma \right),$$

where $\gamma = \gamma^k{}_k$ is the trace of γ and $\partial^2 = \partial^i \partial_i$ is the d'Alembertian.
Hint: Argue immediately that the Γ^2 terms in R_{ij} are 0, without doing any computations with them!

c) The Einstein tensor to first order in γ is

$$G^{(1)}_{ij} = R^{(1)}_{ij} - \frac{1}{2} \eta_{ij} R^{(1)},$$

where $R = R^i{}_i$ is the Ricci scalar. Define

$$\bar{\gamma}_{ij} = \gamma_{ij} - \frac{1}{2}\eta_{ij}\gamma,$$

and substitute this into the Einstein tensor to find

$$G^{(1)}_{ij} = \frac{1}{2}\left(\partial^l\partial_j\bar{\gamma}_{il} + \partial^l\partial_i\bar{\gamma}_{jl}\right) - \frac{1}{2}\partial^2\bar{\gamma}_{ij} - \frac{1}{2}\eta_{ij}\partial^l\partial^k\bar{\gamma}_{kl}.$$

Hints: Show first that

$$\partial^l\partial_j\gamma_{il} + \partial^l\partial_i\gamma_{jl} = \partial^l\partial_j\bar{\gamma}_{il} + \partial^l\partial_i\bar{\gamma}_{jl} + \partial_i\partial_j\gamma.$$

Next, show that

$$\eta_{ij}\partial^l\partial^k\gamma_{kl} = \eta_{ij}\partial^l\partial^k\bar{\gamma}_{kl} + \frac{1}{2}\partial^2\gamma.$$

d) General relativity has gauge transformations, similar to gauge transformations in electromagnetism. Recall that the 4-potential A has a gauge freedom

$$A_\mu \to A_\mu + \partial_\mu\lambda$$

defined by scalar fields λ. Linearized gravity has a gauge freedom

$$\gamma_{ij} \to \gamma_{ij} + \partial_i v_j + \partial_j v_i$$

defined by vector fields $v = v^i\partial_i$. Show that under a gauge transformation defined by v

$$\partial^j\bar{\gamma}_{ij} \to \partial^j\bar{\gamma}_{ij} + \partial^2 v_i.$$

Thus, by solving

$$\partial^2 v_i = -\partial^j\bar{\gamma}_{ij}$$

for v_i, we are free to set

$$\partial^j\bar{\gamma}_{ij} = 0.$$

This is called **Lorenz gauge**.

e) Show that in the Lorenz gauge, the linearized Einstein equation

$$G^{(1)}_{ij} = 8\pi T_{ij}$$

becomes

$$\partial^2\bar{\gamma}_{ij} = -16\pi T_{ij}.$$

Exercise 2. The Newtonian Limit. Assume that $T = \rho \partial_t \otimes \partial_t$, i.e.

$$T_{\mu\nu} = \rho(\partial_t)_\mu (\partial_t)_\nu,$$

where ∂_t is the vector field in the time direction of our coordinate system. Assume this coordinate system to be global. Further assume that time derivatives of $\bar{\gamma}_{\mu\nu}$ are negligible because *the sources are slowly varying*. In this problem, Greek indices go from $0, \dots, 3$ while Latin indices go from $1, \dots, 3$.

a) Show that the result of Exercise 1(e) becomes

$$\nabla^2 \bar{\gamma}_{ij} = 0,$$

where now $i, j \in \{1, 2, 3\}$ and ∇^2 is the usual laplacian on 3-space, while

$$\nabla^2 \bar{\gamma}_{00} = -16\pi\rho.$$

Hint: What are the components $(\partial_t)_\mu$ in a coordinate system? Argue that the form for T implies $T_{\mu\nu} = 0$ unless $\mu = 0$ and $\nu = 0$.

b) Argue that the unique solution of $\nabla^2 \bar{\gamma}_{ij} = 0$ is $\bar{\gamma}_{ij} = 0$.
Hints: Recall the form of the solution of Laplace's equation in spherical coordinates, and notice that it needs to be well-defined at both $r = 0$ and at $r = \infty$. If $\bar{\gamma}_{ij}$ must be a constant, then we can in fact set it to 0 by a gauge transformation.

c) Denote $(\partial_t)_\mu = t_\mu$, and show that our solution for the perturbed metric $\gamma_{\mu\nu}$ is

$$\gamma_{\mu\nu} = -(4t_\mu t_\nu + 2\eta_{\mu\nu})\varphi,$$

where $\varphi = -\frac{1}{4}\bar{\gamma}_{00}$ satisfies Poisson's equation

$$\nabla^2 \varphi = 4\pi\rho.$$

d) The geodesic equation reads

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0.$$

Assume that the 4-velocity of our particle $dx^\mu / d\tau = (1, 0, 0, 0)$, since our particle is moving much slower than the speed of light in the newtonian limit. Thus, show that

$$\frac{d^2 x^\mu}{dt^2} = -\Gamma^\mu_{00} = \partial_\mu \varphi.$$

where we approximate $\tau = t$ and ignore time derivatives of φ. Note that this is exactly the classical equation of motion for a particle in a gravitational potential φ,

$$\mathbf{a} = -\nabla \varphi.$$