Week 13 Worksheet Solutions
Gravitational Radiation

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Exercise 1. No Dipole Radiation. In this problem, you will show that there is no dipole term in the multipole expansion for gravitational radiation; hence, the quadrupole term derived in class is the leading contribution to gravitational radiation.

a) Recall that in electromagnetism, the dipole moment of two charged particles of equal charge q separated by a distance $\mathbf{x}$ is given by

$$\mathbf{p} = q\mathbf{x}.$$

Generalize this to i) n particles of charges q_i with separations $\mathbf{x}_{ij}$ and ii) a continuous charge distribution with charge density $\rho(\mathbf{x})$. What is the electric dipole density $\mathbf{P}$ (where $d = \int \mathbf{P} d^3x$)?

b) Do part (a) for gravitational masses instead of electrically charged particles.

c) Physically, what is the first rate of change $\dot{\mathbf{p}}$ of the gravitational dipole moment?

d) Argue that $\ddot{\mathbf{p}} = 0$; therefore, there is no mass dipole radiation.

e) In electromagnetism, the next strongest form of radiation is due to the magnetic dipole moment: Given a current density $\mathbf{J}(\mathbf{x})$, the magnetization is

$$\mathbf{M} = \frac{1}{2}\mathbf{x} \times \mathbf{J},$$

so that the magnetic dipole moment is[1]

$$\mathbf{m} = \int \mathbf{M} d^3x.$$

Write down the specialization of this general formula to n charged particles of charges q_i moving with velocities $\mathbf{v}_i$.

f) Do part (e) for gravitational masses instead of electrically charged particles.

[1]This follows from performing the multipole expansion of the vector potential $\mathbf{A}$.

g) For the gravitational analog of the magnetic dipole moment, show that $\dot{\mathbf{m}} = \mathbf{0}$; hence, there is no gravitational dipole radiation *at all*.

a) Given n particles, we have

$$\mathbf{p} = \frac{1}{2}\sum_{i,j=1}^{n} \frac{q_i - q_j}{2}\mathbf{x}_{ij},$$

where the factor of $\frac{1}{2}$ comes from the fact that we overcount in the double summation. Given a continuous charge distribution ρ, we have

$$\mathbf{p} = \int \rho(\mathbf{x})\mathbf{x}d^3x,$$

so that

$$\mathbf{P} = \rho(\mathbf{x})\mathbf{x}.$$

Indeed, by expanding the potential

$$V(\mathbf{x}) = \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}d^3x'$$

with (obtained by using the cosine rule and expanding the square root using the binomial expansion for small x'/x)

$$|\mathbf{x} - \mathbf{x}'| \approx x - \hat{x} \cdot \mathbf{x}',$$

we find

$$\begin{aligned} V(\mathbf{x}) \approx& \int \rho(\mathbf{x}') \left(\frac{1}{x} + \hat{x} \cdot \mathbf{x}'\right) d^3x' \\ \approx& \frac{q}{x} + \frac{\hat{x}}{x^2} \cdot \int \rho(\mathbf{x}')\mathbf{x}'d^3x'. \end{aligned}$$

These are the first two terms of the multipole expansion for the scalar potential: The first term is the monopole term; the second is the dipole term we are interested in.

b) We immediately generalize all the formulas to gravitational masses by replacing $q_i \to m_i$ and charge density by mass density. We also have to invert the signs in (i): Since there are no negatively charged masses, an ideal mass dipole is made up of two equal masses, whereas in the electromagnetic case it is made up of two *oppositely charged* particles.

c) We apply the time derivative to

$$\mathbf{p} = \int \rho\mathbf{x}'d^3x'$$

to find

$$\dot{\mathbf{p}} = \int \rho\dot{\mathbf{x}}'d^3x',$$

where we assume the masses are not changing with time. But note that $\rho\dot{\mathbf{x}}'$ is exactly the momentum per unit volume, so that $\dot{\mathbf{p}}$ is the total momentum of the system.

d) $\ddot{\mathbf{p}} = \mathbf{0}$ by conservation of momentum.

e) The current density for a charged particle q_i with velocity $\mathbf{v}_i$ and (time-dependent) space coordinate $\mathbf{x}$ is

$$\mathbf{J}_i(\mathbf{x}') = q_i \mathbf{v}_\mathbf{i} \delta^3(\mathbf{x} - \mathbf{x}').$$

The total current is the sum of these over all i; hence, the magnetic dipole moment is

$$\mathrm{m} = \sum_i \mathbf{m}_i = \tfrac{1}{2} \sum_i q_i \mathbf{x}_i \times \mathbf{v}_i .$$

f) Just replace $q_i \to m_i$.

g) In the discrete case, we have

$$\dot{\mathbf{m}} = \frac{1}{2} \sum m_i \left(\mathbf{v}_i \times \mathbf{v}_i + \mathbf{x}_i \times \mathbf{a}_i \right),$$

where $\mathbf{a}_i$ is the acceleration of particle i and we have again assumed that the masses are constant in time. Now, the first term is 0 automatically. The second term is also 0, because

$$m\mathbf{x} \times \mathbf{a} = \mathbf{L}$$

is the angular momentum which is also conserved for the system.