Week 15 Worksheet
Cosmology

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The Robertson-Walker metric in spherical coordinates is given by

$$ds^2 = -d\tau^2 + a^2(\tau)\begin{cases} d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\varphi^2), & K = 1 \\ d\psi^2 + \psi^2(d\theta^2 + \sin^2\theta d\varphi^2), & K = 0 \\ d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\varphi^2), & K = -1 \end{cases},$$

where $a(\tau) > 0$ is a positive function of proper time τ and K is the sectional curvature.

Exercise 1. In this problem, assume there is no radiation (or other sources of) pressure in the universe.

a) Let ρ be the average mass density of matter in the universe. Use homogeneity and isotropy to argue that for dust (i.e. matter which exerts no pressure)

$$T_{\mu\nu} = \rho u_\mu u_\nu.$$

Hint: The tangent space in spacetime to a point is 4-dimensional, and it splits into a 1-dimensional subspace spanned by u^μ, the unit tangent vector to the worldline of an isotropic observer, and a 3-dimensional subspace spanned by an orthogonal set of unit vectors $\{s_i{}^\mu\}$ tangent to a homogeneous hypersurface passing through the point. Let the time-time component of T be specified by u, and the space-space components by $\{s_i\}$.

b) Argue that the 10 independent equations which arise from Einstein's equation can be reduced to two by homogeneity and isotropy:

$$\begin{aligned} G_{\tau\tau} &= 8\pi\rho \\ G_{**} &= 0, \end{aligned}$$

where (s^μ is any of the $s_i{}^\mu$)

$$\begin{aligned} G_{\tau\tau} &= G_{\mu\nu}u^\mu u^\nu \\ G_{**} &= G_{\mu\nu}s^\mu s^\nu. \end{aligned}$$

Hints: First, argue that the time-space components are 0 and that the space-space components must be the same. Then, project $G_{\mu\nu}$ onto a homogeneous hypersurface and raise an index with the spatial metric. Use homogeneity to argue that the resulting tensor $G^i{}_j$, viewed as a linear map which takes tangent vectors to tangent vectors, must necessarily be a multiple of the identity (by using the spectral theorem for symmetric matrices).

c) Compute the Ricci tensor and Ricci scalar in i) a closed universe (i.e. constant curvature $K = 1$) and ii) in an open universe (constant curvature $K = -1$). The Christoffel symbols and Ricci tensor are

$$\begin{aligned}\Gamma^k_{ij} =& \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) \\ R_{ij} =& \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k{}_{kj} + \Gamma^k_{ij} \Gamma^l{}_{kl} - \Gamma^k_{lj} \Gamma^l_{ki},\end{aligned}$$

where e.g. $\Gamma^k{}_{jk} = \sum_k \Gamma^k_{jk}$ is the contraction.
Hint: Argue that you only need to calculate R_{00} and R_{11}, and use this to limit the number of Christoffel symbols you need to calculate to 6 (which take only 3 different values!).

d) Using

$$G_{ij} = R_{ij} - \frac{1}{2} \eta_{ij} R,$$

show that the differential equations from (b) become

$$\begin{aligned}3\frac{\dot{a}^2}{a^2} =& 8\pi\rho - \frac{3K}{a^2} \\ 3\frac{\ddot{a}}{a} =& - 4\pi\rho.\end{aligned}$$

It turns out that these also hold for the case of flat spacetime ($K = 0$).
Hint: Note that

$$\begin{aligned}R =& - R_{\tau\tau} + 3R_{**} \\ =& - R_{00} + 3a^{-2} R_{11}.\end{aligned}$$

Exercise 2. Hubble's Law. By analyzing the differential equations for a from Exercise 1, show that $\rho > 0$ implies $\ddot{a} > 0$. Thus, derive Hubble's law,

$$\frac{dR}{d\tau} = HR,$$

where R is the distance between two isotropic observers and $H(\tau) = \dot{a}/a$ is Hubble's constant.