Week 3 Worksheet Solutions
Light Cones and Spacetime Diagrams

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Exercise 0. Warm Up.

a) A chicken crosses the road. Is the spacetime interval between the beginning and end of its journey spacelike, timelike, or null?

b) Close your eyes and then open them briefly. What set of points in spacetime do you see during that brief time interval?

a) Since the two points are contained within the light cone of the chicken, they are separated by a timelike interval.

b) Suppose we open our eyes at $t = 0$. The light that gets to the origin of our spatial coordinate system at $t = 0$ must travel along the light cone centered at that point. Now, what we see is light that hits our retina, which takes up some spatial blob near the origin. Light from the lightcones centered at each point on this blob will hit our eyes at $t = 0$. So, at $t = 0$, we see the room we are standing in, but it is made up of a conglomeration of different spacetime points, which will be separated in time depending on the distance light has to travel from the given point to our eyes. You can think of what we see at $t = 0$ as being given by choosing certain points on the conglomeration of the past lightcones which are centered at the different spatial points of our retinal blob. At $t > 0$, before we close our eyes, we can imagine just shifting this picture up in the time direction. For example, if one of the points we see at $t = 0$ is given by (t_1, x_1, y_1, z_1), then at $t > 0$ we won't see this point. Instead, we will see the point $(t + t_1, x_1, y_1, z_1)$, and similarly for all the other points we see at $t = 0$.

Exercise 1. The sun explodes.

a) We sit down to eat lunch (on Earth) four minutes after the explosion. Is the interval between these two events spacelike or timelike? If it is spacelike, find the reference frame (i.e. find the speed of this frame relative to the Sun-Earth rest frame) in which the two events are simultaneous. If it is timelike, find the reference frame in which the two events occur at the same location in space (and happen sequentially).

b) Repeat part (a) if we sit down to eat lunch 10 minutes after the explosion.

a) Since it takes about nine minutes for light to travel from the Sun to the Earth, there is no way for the explosion to affect us at the spacetime location where we eat lunch. It follows that the event must be spacelike. We can label the coordinates of the explosion by $(t_1, x_1) = (-4, -9)$ in the Earth frame, where we measure both time and space in lightminutes; thus, we sit down to eat lunch at $(t_2, x_2) = (0, 0)$ in these coordinates. For an inertial frame which is moving at speed V in the x-direction relative to our frame, we find (using the Lorentz transformation $t' = \gamma(t - Vx)$)

$$\begin{aligned} t_2' &= 0 \\ t_1' &= \gamma(t_1 - Vx_1). \end{aligned}$$

Since we want $t_1' = t_2'$ in the reference frame in which the two events happen simultaneously, we want $V = t_1/x_1$, so $V = \frac{4}{9}c$.

b) In this case the interval is timelike, so we want $x_1' = x_2'$. This time the coordinates of the explosion are $(t_1, x_1) = (-10, -9)$ (where we still set the spacetime point where we eat lunch to the origin). The Lorentz transformation $x' = \gamma(-Vt + x)$ gives

$$\begin{aligned} x_2' &= 0 \\ x_1' &= \gamma(-Vt_1 + x_1). \end{aligned}$$

Since we want $x_1' = x_2'$, we have

$$\begin{aligned} V &= x_1/t_1 \\ &= \frac{10}{9}c. \end{aligned}$$

Exercise 2. Spacetime Shootout. Spaceships B and C, starting at the same location when each of their clocks reads zero, depart from one another with relative velocity $\frac{3}{5}c$. One week later according to B's clocks, B's captain goes berserk and fires a photon torpedo (which travels at the speed of light) at C. Similarly, when the clocks on C read one week, C's captain does the same at B.

a) Draw a two-dimensional spacetime diagram of the events in B's frame.

b) Same as (a) but for C's frame.

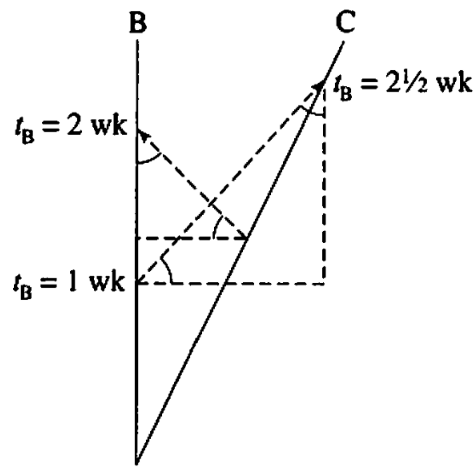
c) Which ship gets hit first? Is there a paradox?

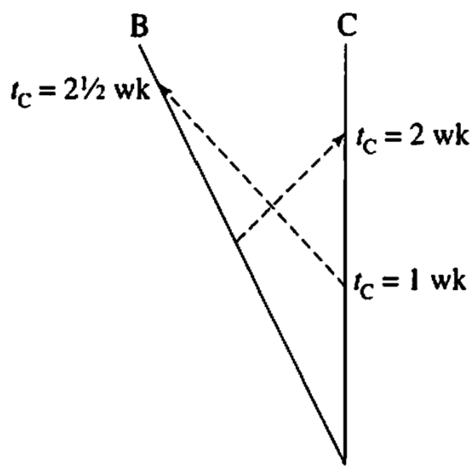
a) See below.

b) See below.

c) In B's frame, B shoots its torpedo after 1 week. To compute when it arrives, we have $t = \frac{3}{5}(1 + t)$, so $t = \frac{5}{2}$ weeks. On the other hand, C's clocks run slow by the γ factor, which in our case is $\gamma = 5/4$. So when $\frac{5}{4}$ weeks passed in B's frame, C shoots a torpedo, and it arrives after $\frac{3}{5} \cdot \frac{5}{4} = \frac{3}{4}$ weeks. So in B's frame, B gets hit 1 week after it sends its torpedo, while C gets hit after 1.5 weeks.

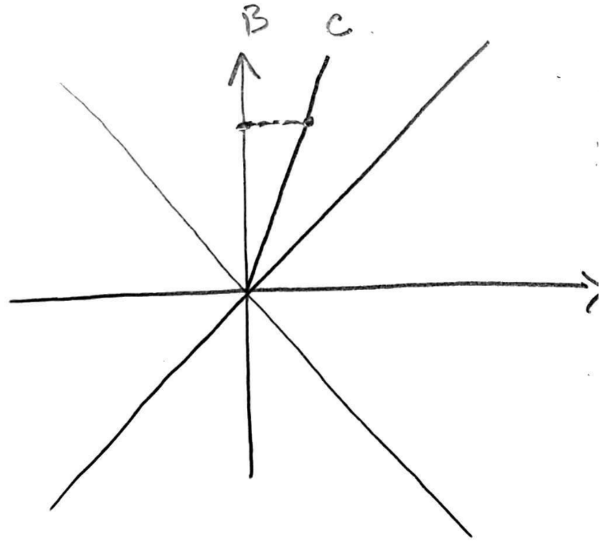
In C's frame, the calculations are identical as in B's frame! Thus, in C's frame, C gets hit before B does. But this is not a paradox: Both B and C get hit *eventually*, just the order of the events in different frames is different.

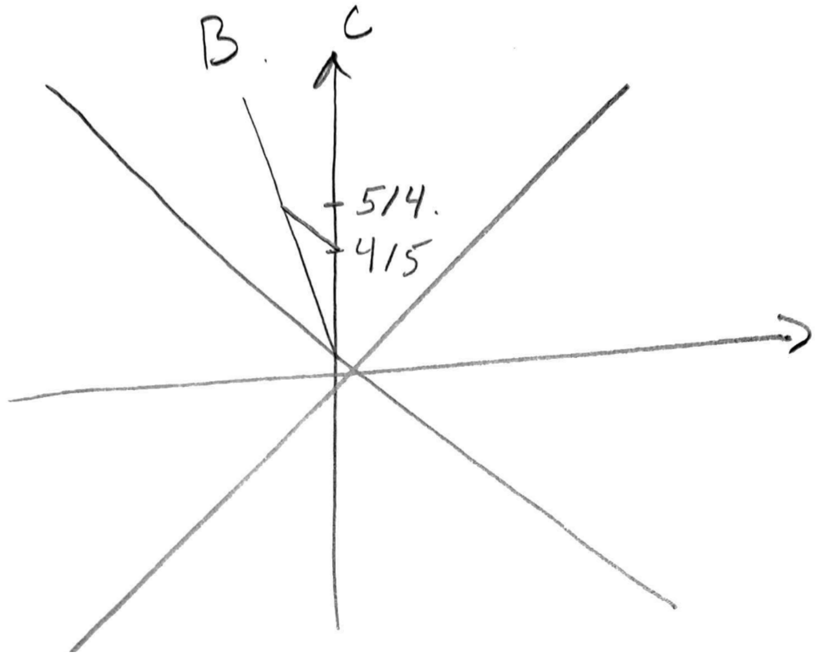




Exercise 3. Time Travel. Repeat Exercise 2, but this time, instead of a photon torpedo, B's captain sends a tachyonic message *at infinite speed* to C (and C's captain does nothing). Draw both spacetime diagrams including the tachyon paths and lightcones (centered on the point of departure). Comment on the shape of the tachyon path in C's frame.

In B's frame, a week passes when the tachyon is sent out. This means C's clocks read $4/5$ weeks when it arrives (since it arrives and is sent out simultaneously). Now, in C's frame, B's clocks run slow. So it sends out the tachyon at $5/4$ weeks, which means it arrives *before* it gets sent in C's frame!


B
C


B
C
5/4
4/5