Week 4 Worksheet Solutions
4-vectors, Energy-Momentum, and the Metric

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Exercise 1.

a) Show that the squared length of any particle's four-velocity is -1.

b) Show that

$$m^2 = E^2 - p^2$$

in natural units (i.e. in units such that $c = 1$).
Hint: The p^2 in this formula is *not* the squared length of the 4-momentum p; rather, it is the squared length of the 3-vector $\mathbf{p}$!

a) We have

$$u^2 = \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\eta_{\mu\nu} = -\left(\frac{dx^0}{d\tau}\right)^2 + \sum\left(\frac{dx^i}{d\tau}\right)^2 = -\gamma + \gamma v^2 = -1.$$

b) By definition, $p = mu$, where u is the 4-velocity. By (a),

$$p^2 = (mu)^2 = m^2u^2 = -m^2.$$

Likewise,

$$p^2 = m^2\left[-(u^0)^2 + u^i u_i\right] = -m^2\gamma + m^2|\mathbf{u}|^2\gamma = E^2 + |\mathbf{p}|^2$$

Exercise 2. A particle of rest mass m and 4-momentum p is examined by an observer with 4-velocity u. Show the following statements.

a) The energy the observer measures is

$$E = -p \cdot u.$$

b) The rest mass the observer attributes to the particle is

$$m^2 = -p^2.$$

c) The 3-momentum the observer measures has magnitude

$$|\mathbf{p}'| = \sqrt{(p \cdot u)^2 + p^2}.$$

d) The 3-velocity $\mathbf{v}$ the observer measures has magnitude

$$|\mathbf{v}| = \frac{|\mathbf{p}'|}{E}.$$

e) The 4-vector v ("ordinary velocity"), whose components in the observer's frame are

$$v^0 = 0, \quad v^j = dx^j/dt,$$

is given by

$$v = \frac{p + (p \cdot u)u}{-p \cdot u}.$$

Here, t is understood to be the time *for the particle*.

a) **First Solution:** We can choose our coordinate axes so that $p = (p^0, p^1, 0, 0)$ and $u = (u^0, u^1, u^2, 0)$. In the observer's frame, the energy is given by

$$p'^0 = \gamma p^0 - \gamma u^1 p^1,$$

since the component of u in the direction of $\mathbf{p}$ is u^1. On the other hand,

$$-p \cdot u = p^0 u^0 - p^1 u^1 = p'^0.$$

Second (*Fancy*) **Solution**: In the rest frame of the particle, this formula holds, since $E = p^0 = p^0 u^0$. Now, suppose the particle has some velocity relative to the observer. In the observer's frame, then, $u = (1, \mathbf{0})$ still. Then $p \cdot u = -p^0 = -E$.

b) This follows by the definition of $p = mv$ and the fact that $v^2 = -1$.

c) We have

$$|\mathbf{p}'|^2 = p'^2 + (p'^0)^2 = p'^2 + (p \cdot u)^2$$

by (a). Since $p^2 = p'^2$ because length is invariant under Lorentz transformations, we have the result.

d) This follows from the relation $\mathbf{p}' = m\gamma\mathbf{v}$ and the fact that $E = m\gamma$.

e) Note that in the observer's frame, $p^0 = E = Eu^0$, since $u^0 = 1$ in that frame. Then

$$\frac{p^0 - E}{E} = \frac{E - E}{E} = 0$$

as desired. We then have

$$\frac{p^i - Eu^i}{E} = v^i$$

since $p^i = Ev^i$ and $u^i = 0$.