Week 6 Worksheet
Equivalence Principle

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Exercise 0. Warm up.

a) While sitting in a car at a stoplight, you're holding a helium-filled balloon on a string. The light turns green and the car accelerates. What happens to the balloon? Give a physical explanation of your result *without using the equivalence principle*!

b) A balloon filled with helium is released at rest in the middle of a drifting spaceship in gravity-free space. The spaceship is filled with air. What does the balloon do?

c) The spaceship accelerates forward. Does the balloon drift to the front or back of the rocket?

d) You are holding a cup of hot tea in a coasting car. The driver steps on the brakes. Which way should you tip the cup so its contents won't spill out?

e) A car is at rest with one of its doors open. Describe how to close the door by accelerating the car.

a) The equivalence principle says that the balloon will tilt forward in the car. But we can also see this from more basic principles. As the car accelerates, the air pressure in the back of the car increases, while in the front of the car it will decrease. Since helium is lighter than air, it will be pushed by the gradient in air pressure towards the location of higher pressure, hence towards the front of the car.

b) Since there is no acceleration nor pressure gradients, the balloon will stay in place.

c) Now a pressure gradient appears, so the balloon will move toward the front of the rocket.

d) You should tilt it away from yourself.

e) Accelerate forward.

Exercise 1. Find the altitude of a satellite in a circular orbit about the Earth whose clocks run *at the same rate* as Earth clocks. The radius of the Earth is 6.4 km.
Hints: You can determine the value of GM_E by noting that $g = GM_E/R_E$. Also recall that the ratio of the rate signals are received at A to the rate signals are emitted at B is

$$1 + \Phi_A - \Phi_B.$$

There are two competing effects here, one given by time dilation due to special relativity and the other time contraction due to general relativity. The ratio of the rates mentioned in the hint is the same as $\frac{1}{\gamma}$, since the rate signals are received at A is $1/\Delta t_A$, while the rate signals are emitted at B is $1/\Delta\tau_B$. Thus,

$$\frac{1/\Delta t_A}{1/\Delta\tau_B} = \frac{1}{\gamma}.$$

Since the satellite is moving nonrelativistically, we can expand the $1/\gamma$ factor as

$$1/\gamma \approx 1 - V^2/2.$$

Note that this is the γ factor of the satellite relative to the Earth. Since the velocities are slow, the relative velocity is just $V_s - V_e$ plus corrections of order V^2/c^2 which we can ignore. You can calculate the speed $\frac{8}{9}$ km/s at the surface of the Earth by using the fact that the surface makes a 2π rotation in 24 hours. Now, in a circular orbit we have

$$\frac{V_s^2}{R} = \frac{GM}{R^2},$$

so

$$V_s^2 = \frac{GM}{R}.$$

So the correction $V^2/2$ from special relativity must cancel out the correction $\Delta\Phi$ due to gravity. Now,

$$\Delta\Phi = \frac{GM}{R_e} - \frac{GM}{R}.$$

Setting this equal to V^2, we find

$$\left(\sqrt{\frac{gR_e^2}{R}} - V_e\right)^2 = gR_e(1 - \frac{R_e}{R}).$$

This is a quadratic equation in $\sqrt{R}$ and hence can be solved for $\sqrt{R}$ analytically.

Exercise 2. Life on a Cylinder. Consider a cylinder of radius r in ordinary euclidean 3-space. Write down the metric on the cylinder induced from the euclidean one on 3-space as follows.

a) Model the cylinder as $Z = S_r^1 \times \mathbb{R}$, where S_r^1 denotes the circle of radius r. There is a map from the cylinder minus a line to $\mathbb{R}^2$, $Z \setminus \ell \to \mathbb{R}^2$, where we cut out a line ℓ along the "vertical" direction of the cylinder and then "roll" the cylinder out onto the plane. Argue that this map is an **isometry**, so that it gives an equivalence between the metric on the cylinder minus a line and the metric on $\mathbb{R}^2$.

b) Part (a) tells you that the shortest path between two points will be one which subtends an angle on the circle S_r^1 which is not greater than π. Given any two points, we can now choose which line to cut out of the cylinder to determine the distance between them. Using this fact and the isometry in (a), compute the metric.

a) Label points on the cylinder minus a line by (θ, h), where $\theta \in (-\pi, \pi)$ is the angle of the point (0 corresponds to the line opposite to ℓ) and $h \in \mathbb{R}$ is the height of the point. Then the map in question sends $(\theta, h) \mapsto (r\theta, h) \in \mathbb{R}^2$. Note that $\theta = 0$ corresponds to the y-axis in the plane. It's immediately clear now that the metric must be given by $ds^2 = dh^2 + r^2 d\theta^2 + \alpha(h, \theta) dh d\theta$, where you can obtain the first two terms by considering distances that are only in h or only in θ and α is some function of h and θ. We just need to show that this cross term is 0. This should be intuitively obvious, but I've included the detailed mathematical proof below. Please ignore it if you don't understand the notation! We just need to check that

$$i^* \langle v, w \rangle_{\mathbb{R}^3} = f^* \langle v, w \rangle_{\mathbb{R}^2}$$

for $i : Z \hookrightarrow \mathbb{R}^3$ the inclusion, f the map $Z \setminus \ell \to \mathbb{R}^2$, $\langle -, - \rangle_{\mathbb{R}^3}$ the euclidean metric on $\mathbb{R}^3$, and $\langle -, - \rangle_{\mathbb{R}^2}$ the euclidean metric on the plane. Here, v, w are tangent vectors (located at the same point p) on the cylinder. The LHS is just the ordinary scalar product $v^i w_i$, and we need to check this for

$$v = \frac{\partial}{\partial h} = (0, 1)$$
$$w = \frac{\partial}{\partial \theta} = (1, 0).$$

Note that

$$f_* v = v(f)^i \partial_i = (0, 1),$$

and

$$f_* w = w(f)^i \partial_i = (r, 0),$$

where we use the notation (a, b) to denote a tangent vector

$$a\partial_x + b\partial_y$$

on the plane. Thus,

$$f^* \langle v, w \rangle = \langle f_* v, f_* w \rangle = (0, 1) \cdot (r, 0) = (f_* v)^i (f_* w)_i = 0 = v^i w_i$$

as desired.

b) So now given two points we can effectively just compute their distance after we roll out the cylinder onto $\mathbb{R}^2$. But this is just $\sqrt{h^2 + (\theta r)^2}$, where $h = |h_1 - h_2|$ and $\theta = |\theta_1 - \theta_2|$ are the height and angle differences between the points (θ_i, h_i). So the metric is

$$ds^2 = dh^2 + r^2 d\theta^2.$$