Week 7 Worksheet
Geodesics and Covariant Derivatives

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Exercise 1. Reparametrization of Geodesics. If γ is a geodesic parametrized by proper time, the geodesic equation reads

$$\frac{d^2\gamma^k}{d\tau^2} + \Gamma^k_{ij}(\gamma(\tau))\frac{d\gamma^i}{d\tau}\frac{d\gamma^j}{d\tau} = \frac{d\gamma^i}{d\tau}\nabla_i\frac{d\gamma^k}{d\tau} = 0.$$

a) Show that if c is a geodesic parametrized by proper time and $\tau(t)$ is a reparametrization, so that $\gamma(t) = c(\tau(t))$ is not parametrized by proper time, then we need to modify this equation to instead read

$$\frac{d^2\gamma^k}{dt^2} + \Gamma^k_{ij}(\gamma(t))\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = \frac{d\gamma^k}{dt}\frac{\tau''(t)}{\tau'(t)}. \tag{1}$$

Hints: Use the fact that geodesics are length-minimizing paths and the Euler-Lagrange equations for the length functional

$$F(\gamma, \dot{\gamma}) = \left\| \frac{d\gamma}{dt} \right\|.$$

The formula for the Christoffel symbols Γ^k_{ij} is given in Exercise 2a.

b) Conversely, if $\gamma(t) = c(\tau(t))$ satisfies Equation 1, show that c is a geodesic.

c) In fact, if γ satisfies

$$\frac{d^2\gamma^k}{dt^2} + \Gamma^k_{ij}(\gamma(t))\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = \frac{d\gamma^k}{dt}\mu(t),$$

for μ any function of t, show that γ is also a reparametrization of a geodesic. Note that in ∇ notation this reads

$$\frac{d\gamma^i}{dt}\nabla_i\frac{d\gamma^k}{dt} = \frac{d\gamma^k}{dt}\mu(t).$$

Remark. Note that part (c) shows that we don't lose anything by always considering geodesics which are parametrized by proper time. This is called **parametrization by arclength** or, alternatively, **affine parametrization**.

a) We have the length functional as stated in the hint $F(\gamma, \dot{\gamma})$. We want to minimize the integral of this, so we compute

$$\begin{aligned}\frac{\partial F}{\partial \gamma^\ell} &= \frac{1}{2F}\frac{\partial g_{ij}}{\partial \gamma^\ell}\dot{\gamma}^i\dot{\gamma}^j \\ \frac{\partial F}{\dot{\gamma}^\ell} &= \frac{1}{F}g_{\ell j}\dot{\gamma}^j.\end{aligned}$$

So

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{\gamma}^\ell} = -\frac{g_{\ell j}\dot{\gamma}^j}{F^2}\dot{F} + \frac{\partial g_{\ell j}}{\partial \gamma^i}\dot{\gamma}^i\dot{\gamma}^j + g_{\ell j}\ddot{\gamma}^j.$$

Thus, Euler-Lagrange reads

$$g_{\ell j}\dot{\gamma}^j\frac{\dot{F}}{F} = \frac{\partial g_{\ell j}}{\partial \gamma^i}\dot{\gamma}^i\dot{\gamma}^j + g_{\ell j}\ddot{\gamma}^j - \frac{1}{2}\frac{\partial g_{ij}}{\partial \gamma^\ell}\dot{\gamma}^i\dot{\gamma}^j.$$

Apply $g^{k\ell}$ to both sides (summing over ℓ) to find

$$\dot{\gamma}^k\frac{\dot{F}}{F} = \frac{1}{2}g_{k\ell}\left(\frac{\partial g_{\ell j}}{\partial \gamma^i} + \frac{\partial g_{\ell j}}{\partial \gamma^j} - \frac{\partial g_{ij}}{\partial \gamma^\ell}\right)\dot{\gamma}^i\dot{\gamma}^j + \ddot{\gamma}^k.$$

Now, notice that the first term on the RHS is exactly $\Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j$. We're almost done; it remains to note that $F = \tau'$ and $\dot{F} = \tau''$. Indeed, if $s(t) = L_a^t(\gamma)$ is arclength, then

$$\frac{ds}{dt} = \left\|\frac{d\gamma}{dt}\right\| = \|\dot{c}\tau'\| = \tau',$$

since c is parametrized by arclength and we assume $\tau' > 0$ is the tangent to τ along t. Similarly, $\ddot{s} = \tau''$.

b) We check

$$\ddot{\gamma}^k = \frac{d^2}{dt^2}(c^k \circ \tau) = \frac{d}{dt}\left(\dot{c}^k\frac{d\tau}{dt}\right) = \dot{c}^k\tau'' + \ddot{c}^k \cdot (\tau')^2.$$

On the other hand,

$$\dot{\gamma}^i = \dot{c}^i\tau'.$$

Combining the two equations we find

$$\dot{c}^k\tau'' + \ddot{c}^k(\tau')^2 + \Gamma^k_{ij}\dot{c}^i\dot{c}^j(\tau')^2 = \dot{c}^k\tau''.$$

But this reduces to the geodesic equation for c parametrized by proper time.

c) This follows immediately if we find a function τ such that $\tau''/\tau' = \mu$. But this differential equation can be straightforwardly solved:

$$\frac{d\tau'}{\tau'} = \mu dt$$

can be integrated to obtain

$$\tau = \int_0^t \exp\left(\int_0^{t'} \mu dt''\right) dt'.$$

Exercise 2. The Poincaré Upper Half-plane. Consider the **Poincaré upper half-plane** $\mathbb{H}^2 = \{(x, y) | y > 0\}$ with the metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2).$$

a) Compute the Christoffel symbols

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

for this metric.

b) Consider semicircles in $\mathbb{H}^2$ centered on the x-axis of radius R. Model the semicircles as curves $c(t) = (t, \gamma(t))$, and show that

$$\gamma''(t) = -\frac{\gamma'(t)}{t - c} - \frac{\gamma'(t)^2}{\gamma(t)}.$$

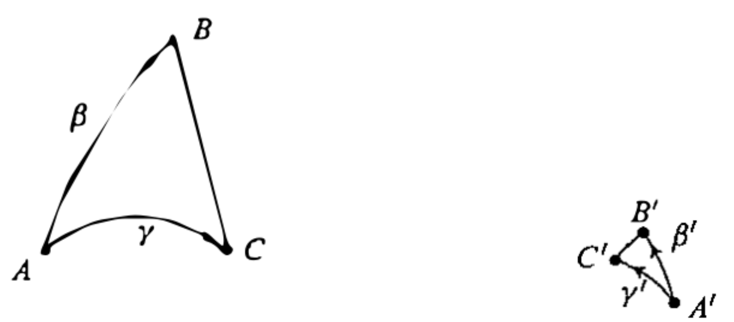
c) Use Exercise 1 to show that in $\mathbb{H}^2$ the (suitably parametrized) semicircles with center on the x-axis and all straight lines parallel to the y-axis are geodesics. In fact, it can be shown that these are *all* the geodesics, but don't bother to do this.

d) Check that all geodesics have infinite length in either direction. We say that $\mathbb{H}^2$ is **complete**.

e) **Optional Challenge**: If you know about conformal mappings, try this problem. Consider the mappings $f(z) = \frac{az+b}{cz+d}$, where $ad - bc > 0$, a, b, c, d are real, and we consider the upper half-plane $\mathbb{H}^2$ as a subset of the complex plane $\mathbb{C}$ with complex coordinate $z = x + iy$. Show that f is an isometry and that given any tangent vector $v = v^i \partial/\partial x^i|_p$ at one point and any tangent vector $w = w^i \partial/\partial x^i|_q$ at any other point, f_* maps one to the other for some values of a, b, c, d. Here, $f_* v(g) = v(g(f(p)))$. Use this to formulate and prove a version of the "side-angle-side" theorem for **triangles**—three-sided figures whose sides are all geodesics. These results show that $\mathbb{H}^2$ is a model for lobachevskian non-euclidean geometry; the sum of the angles in any triangle is $< \pi$.

Remark. Note that the sum of the angles in any triangle on the sphere is $> \pi$. We say that the sphere has positive curvature, while the Poincaré upper half-plane $\mathbb{H}^2$ has negative curvature (of course, flat space has 0 curvature). These notions will be made precise in the coming weeks.

I'm going to leave out the solution to this exercise, since you can use it for review for the final. The parts that are on the midterm review will have solutions posted along with the midterm review solutions.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