Week 7 Worksheet
Geodesics and Covariant Derivatives

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Exercise 1. Reparametrization of Geodesics. If γ is a geodesic parametrized by proper time, the geodesic equation reads

$$\frac{d^2\gamma^k}{d\tau^2} + \Gamma^k_{ij}(\gamma(\tau))\frac{d\gamma^i}{d\tau}\frac{d\gamma^j}{d\tau} = \frac{d\gamma^i}{d\tau}\nabla_i\frac{d\gamma^k}{d\tau} = 0.$$

a) Show that if c is a geodesic parametrized by proper time and $\tau(t)$ is a reparametrization, so that $\gamma(t) = c(\tau(t))$ is not parametrized by proper time, then we need to modify this equation to instead read

$$\frac{d^2\gamma^k}{dt^2} + \Gamma^k_{ij}(\gamma(t))\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = \frac{d\gamma^k}{dt}\frac{\tau''(t)}{\tau'(t)}. \tag{1}$$

Hints: Use the fact that geodesics are length-minimizing paths and the Euler-Lagrange equations for the length functional

$$F(\gamma, \dot{\gamma}) = \left\| \frac{d\gamma}{dt} \right\|.$$

The formula for the Christoffel symbols Γ^k_{ij} is given in Exercise 2a.

b) Conversely, if $\gamma(t) = c(\tau(t))$ satisfies Equation 1, show that c is a geodesic.

c) In fact, if γ satisfies

$$\frac{d^2\gamma^k}{dt^2} + \Gamma^k_{ij}(\gamma(t))\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = \frac{d\gamma^k}{dt}\mu(t),$$

for μ any function of t, show that γ is also a reparametrization of a geodesic. Note that in ∇ notation this reads

$$\frac{d\gamma^i}{dt}\nabla_i\frac{d\gamma^k}{dt} = \frac{d\gamma^k}{dt}\mu(t).$$

Remark. Note that part (c) shows that we don't lose anything by always considering geodesics which are parametrized by proper time. This is called **parametrization by arclength** or, alternatively, **affine parametrization**.

Exercise 2. The Poincaré Upper Half-plane. Consider the **Poincaré upper half-plane** $\mathbb{H}^2 = \{(x, y) | y > 0\}$ with the metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2).$$

a) Compute the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

for this metric.

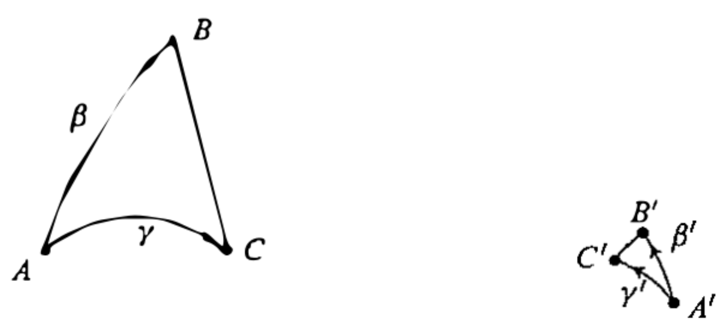
b) Consider semicircles in $\mathbb{H}^2$ centered on the x-axis of radius R. Model the semicircles as curves $c(t) = (t, \gamma(t))$, and show that

$$\gamma''(t) = -\frac{\gamma'(t)}{t - c} - \frac{\gamma'(t)^2}{\gamma(t)}.$$

c) Use Exercise 1 to show that in $\mathbb{H}^2$ the (suitably parametrized) semicircles with center on the x-axis and all straight lines parallel to the y-axis are geodesics. In fact, it can be shown that these are *all* the geodesics, but don't bother to do this.

d) Check that all geodesics have infinite length in either direction. We say that $\mathbb{H}^2$ is **complete**.

e) **Optional Challenge**: If you know about conformal mappings, try this problem. Consider the mappings $f(z) = \frac{az+b}{cz+d}$, where $ad - bc > 0$, a, b, c, d are real, and we consider the upper half-plane $\mathbb{H}^2$ as a subset of the complex plane $\mathbb{C}$ with complex coordinate $z = x + iy$. Show that f is an isometry and that given any tangent vector $v = v^i \partial/\partial x^i|_p$ at one point and any tangent vector $w = w^i \partial/\partial x^i|_q$ at any other point, f_* maps one to the other for some values of a, b, c, d. Here, $f_* v(g) = v(g(f(p)))$. Use this to formulate and prove a version of the "side-angle-side" theorem for **triangles**—three-sided figures whose sides are all geodesics. These results show that $\mathbb{H}^2$ is a model for lobachevskian non-euclidean geometry; the sum of the angles in any triangle is $< \pi$.



Remark. Note that the sum of the angles in any triangle on the sphere is $> \pi$. We say that the sphere has positive curvature, while the Poincaré upper half-plane $\mathbb{H}^2$ has negative curvature (of course, flat space has 0 curvature). These notions will be made precise in the coming weeks.