Week 9 Worksheet
Curvature

Jacob Erlikhman

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Exercise 1. Parallel Transport is Curvature. Let M be a spacetime such that for any two points $p, q \in M$, the parallel transport from p to q does not depend on the curve that joints p and q. You will show that this implies that M is flat, i.e. that the Riemann curvature tensor on M is identically 0. We will do this with the help of the following construction. Consider a parametrized surface $f : U \to M$, where

$$U = \{(s,t) \in \mathbb{R}^2 | s, t \in (-\varepsilon, 1+\varepsilon),\ \varepsilon > 0\}$$

and we force $f(s,0) = f(0,0)$ for all s. Let V_0 be a tangent vector to M at $f(0,0)$, and define a vector field V along f as follows. Set $V(s,0) = V_0$ and $V(s,t)$ to be the parallel transport of V_0 along the curve $c(t) = f(s,t)$.

a) Sketch V in the case that M is flat, and explain what changes in the non-flat case.

b) Argue that we can assume that all the curves $c(t)$ for fixed s are parametrized by proper time $t = \tau$.

c) Since V is parallel transported along the t-direction, what is $\nabla_{\partial_t f} V$?

d) Recall that Riemann curvature is a rank (3,1) tensor R which in a coordinate system x^i is given by

$$R(\partial_j, \partial_k)Z = -Z^l R^i{}_{ljk}\partial_i,$$

where $Z = Z^i\partial_i = Z^i\frac{\partial}{\partial x^i}$ is a vector field. Write down an analogous formula for $R(X,Y)Z$.
Hint: Recall that tensors are linear *in functions* in each of their inputs!

e) Now, use the Ricci identity

$$-Z^l R^i{}_{ljk} = \nabla_j\nabla_k Z^i - \nabla_k\nabla_j Z^i$$

to show that

$$\nabla_{\partial_t f}\nabla_{\partial_s f} V + R(\partial_s f, \partial_t f)V = 0.$$

Hints: Write out $\partial_s f$ and $\partial_t f$ (and V) in a coordinate system. Then, use the formula from (d) and linearity of $R(X,Y)Z$. Note that if $\partial_s f = X^i\partial_i$ and $\partial_t f = Y^i\partial_i$, then

$$\nabla_{\partial_s f} Y^i = \nabla_{\partial_t f} X^i,$$

since $\partial_s\partial_t f = \partial_t\partial_s f$.

f) Show that $V(s,1)$ is also the parallel transport of $V(0,1)$ along the curve $c(s) = f(s,1)$, so that $\nabla_{\partial_s f} V(s,1) = 0$.

g) Show that

$$R(\partial_s f, \partial_t f)V(0,1) = 0,$$

where the $(0,1)$ means we consider the vector at the point $(s,t) = (0,1)$.

h) Conclude that $R = 0$ everywhere by arbitrariness of our choices.

Remark. There is another way to solve (e) which uses the invariant definition of the Riemann curvature,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

We simply compute

$$R(\partial_s f, \partial_t f)V = \nabla_{\partial_s f} \nabla_{\partial_t f} V - \nabla_{\partial_t f} \nabla_{\partial_s f} V + \nabla_{[\partial_s f, \partial_t f]} V.$$

The first term vanishes because $\nabla_{\partial_t f} V = 0$. The last term vanishes because $[\partial_s f, \partial_t f] = [f_* \partial_s, f_* \partial_t] = f_*[\partial_s, \partial_t] = 0$ because the partial derivatives ∂_s and ∂_t commute on $\mathbb{R}^2$. This gives the result.

a) Fix a vector V_0. Since it's parallely transported along the t-direction and parallel transport in flat space is the same as just setting $V(0,t) = V_0$ for all t, we know what V is along the t-axis of the surface in M. Similarly, $V(s,0) = V_0$ by definition. It remains to see what happens in between, but this will just be given by the parallel transport of V_0 along $c_s(t) = f(s,t)$ for fixed s, hence the vector $V(s,t) = V_0$ for all s and t. When M is not flat, the parallel transport of V_0 along $c_0(t) = f(0,t)$ will in general be different than V_0; there will be some rotation, i.e. $V(0,t)$ will in general be a rotated version of V_0. On the other hand $V(s,0) = V_0$ by definition. Note that the vector field along the curve $c_0(t)$ will in general be different than the vector field along the curve $c_s(t)$ for $s \neq 0$, since the sectional curvature will in general be different at different points in the image of f.

b) The parametrization to begin with was arbitrary, so we're free to reparametrize.

c) Since parallel transport of a vector field V along a curve $c(t)$ is by definition the solution to the differential equation

$$\nabla_{\dot{c}(t)} V = 0$$

with initial condition $V\,(t = 0) = V_0$, and since (for fixed s) $\dot{c}(t) = \partial_t f$, we have

$$\nabla_{\partial_t f} V = 0.$$

d) Since R is linear, we can write $X = X^i \partial_i$, $Y = Y^i \partial_i$, and thus

$$R(X,Y)Z = -X^j Y^k Z^l R^i{}_{ljk} \partial_i.$$

e) We plug in the Ricci identity to our result for (d) to find

$$R(X,Y)Z = X^j Y^k (\nabla_j \nabla_k Z^i - \nabla_k \nabla_j Z^i).$$

Now, plug in $X = \partial_s f$, $Y = \partial_t f$, and $Z = V$ and use linearity of R again together with the statement in the hint to find

$$R(\partial_s f, \partial_t f)V = \nabla_{\partial_s f} \nabla_{\partial_t f} V - \nabla_{\partial_t f} \nabla_{\partial_s f} V.$$

But the first term vanishes by (c), so we're done. More explicitly, we obtain the above formula by writing

$$R(X,Y)V = X^j Y^k (\nabla_j \nabla_k V^i - \nabla_k \nabla_j V^i)\partial_i.$$

To bring the X's and Y's into the derivatives, we need to check that

$$X^j \nabla_j Y^k = Y^j \nabla_j X^k,$$

but this follows by the statement given in the hint, since e.g. $X^j \nabla_j = \nabla_X$.

f) Parallel transport doesn't depend on the curve chosen to get to the desired point. Since we can choose the curve which first goes along t to get to $f(0,1)$ and then along s to get to $f(1,1)$, this is the same as just going directly along t to get to $f(1,1)$.

g) This follows by combining (e) with (f).

h) We could've chosen any points $p = f(0,0)$ and $q = f(0,1)$ in the previous. Thus, we find that the Riemann curvature tensor will vanish at q for the choices of V and f. But f was arbitrary, so so are $\partial_t f$ and $\partial_s f$. Similarly, V_0 was arbitrary, so R vanishes identically at q. Since q is arbitrary, we're done.