Week 9 Worksheet
Curvature

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Exercise 1. Parallel Transport is Curvature. Let M be a spacetime such that for any two points $p, q \in M$, the parallel transport from p to q does not depend on the curve that joints p and q. You will show that this implies that M is flat, i.e. that the Riemann curvature tensor on M is identically 0. We will do this with the help of the following construction. Consider a parametrized surface $f : U \to M$, where

$$U = \{(s,t) \in \mathbb{R}^2 | s, t \in (-\varepsilon, 1+\varepsilon),\ \varepsilon > 0\}$$

and we force $f(s,0) = f(0,0)$ for all s. Let V_0 be a tangent vector to M at $f(0,0)$, and define a vector field V along f as follows. Set $V(s,0) = V_0$ and $V(s,t)$ to be the parallel transport of V_0 along the curve $c(t) = f(s,t)$.

a) Sketch V in the case that M is flat, and explain what changes in the non-flat case.

b) Argue that we can assume that all the curves $c(t)$ for fixed s are parametrized by proper time $t = \tau$.

c) Since V is parallel transported along the t-direction, what is $\nabla_{\partial_t f} V$?

d) Recall that Riemann curvature is a rank (3,1) tensor R which in a coordinate system x^i is given by

$$R(\partial_j, \partial_k)Z = -Z^l R^i{}_{ljk} \partial_i,$$

where $Z = Z^i \partial_i = Z^i \frac{\partial}{\partial x^i}$ is a vector field. Write down an analogous formula for $R(X,Y)Z$.
Hint: Recall that tensors are linear *in functions* in each of their inputs!

e) Now, use the Ricci identity

$$-Z^l R^i{}_{ljk} = \nabla_j \nabla_k Z^i - \nabla_k \nabla_j Z^i$$

to show that

$$\nabla_{\partial_t f} \nabla_{\partial_s f} V + R(\partial_s f, \partial_t f)V = 0.$$

Hints: Write out $\partial_s f$ and $\partial_t f$ (and V) in a coordinate system. Then, use the formula from (d) and linearity of $R(X,Y)Z$. Note that if $\partial_s f = X^i \partial_i$ and $\partial_t f = Y^i \partial_i$, then

$$\nabla_{\partial_s f} Y^i = \nabla_{\partial_t f} X^i,$$

since $\partial_s \partial_t f = \partial_t \partial_s f$.

f) Show that $V(s,1)$ is also the parallel transport of $V(0,1)$ along the curve $c(s) = f(s,1)$, so that $\nabla_{\partial_s f} V(s,1) = 0$.

g) Show that

$$R(\partial_s f, \partial_t f)V(0,1) = 0,$$

where the $(0,1)$ means we consider the vector at the point $(s,t) = (0,1)$.

h) Conclude that $R = 0$ everywhere by arbitrariness of our choices.

Remark. There is another way to solve (e) which uses the invariant definition of the Riemann curvature,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

We simply compute

$$R(\partial_s f, \partial_t f)V = \nabla_{\partial_s f} \nabla_{\partial_t f} V - \nabla_{\partial_t f} \nabla_{\partial_s f} V + \nabla_{[\partial_s f, \partial_t f]} V.$$

The first term vanishes because $\nabla_{\partial_t f} V = 0$. The last term vanishes because $[\partial_s f, \partial_t f] = [f_* \partial_s, f_* \partial_t] = f_*[\partial_s, \partial_t] = 0$ because the partial derivatives ∂_s and ∂_t commute on $\mathbb{R}^2$. This gives the result.