# The Cotangent Complex

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## 1 Introduction

Geometric deformation problems arise in many contexts. For example, one might be interested in a moduli problem or a quantization problem. Given a complex symplectic manifold, deformations of a lagrangian submanifold $Y$ are controlled (in the differential or algebraic geometric setting) by elements in $T^*Y$. The key is an identification $T^*Y \cong NY$ via the symplectic form. In the more general setting of schemes, we'd like to deform schemes themselves, or possibly morphisms of schemes. In other words, we'd like to solve the following

**Problem 0. Baby deformations.** Fix a commutative diagram of schemes

$$\begin{array}{ccc} X_0 & & \\ \big\downarrow{\scriptstyle f_0} & & \\ Y_0 & \overset{j}{\hookrightarrow} & Y \\ \big\downarrow & \swarrow & \\ S & & \end{array}, \tag{1}$$

where $X_0, Y_0$, and $Y$ are thought of as schemes over $S$. We'd like to complete such a diagram to

$$\begin{array}{ccc} X_0 & \overset{i}{\hookrightarrow} & X \\ \big\downarrow{\scriptstyle f_0} & & \big\downarrow \\ Y_0 & \overset{j}{\hookrightarrow} & Y \\ \big\downarrow & \swarrow & \\ S & & \end{array}. \tag{2}$$

Here, $Y$ is an "infinitesimal thickening" or "square-zero extension" of $Y_0$, and we'd like to find another "square-zero extension" of $X_0$ in a compatible manner to $Y_0 \subset Y$. In fact, we can consider the much more general situation.

**Problem 1. Scheme deformations.** Suppose in (1) we instead have $X_0$ flat over $Y_0$ (and locally of finite presentation; this is a technical condition), so we want to deform *an entire family of schemes*. I.e. we want to find $X$ flat over $Y$ making the diagram commute.

Finally, rather than deforming schemes we may want to deform scheme morphisms. Namely, we want to consider

**Problem 2. Morphism deformations.** Suppose we have the following commutative diagram of schemes,

$$\begin{array}{ccc} X_0 & \overset{i}{\hookrightarrow} & X \\ \downarrow{\scriptstyle f_0} & & \\ Y_0 & \overset{j}{\hookrightarrow} & Y \\ \downarrow & & \downarrow \\ Z_0 & \overset{k}{\hookrightarrow} & Z \\ \downarrow & \swarrow & \\ S & & \end{array}, \qquad (h_0 : X_0 \to Z_0,\ X \to Z) \tag{3}$$

where $X, Y, Z$ are "square-zero extensions" of $X_0, Y_0, Z_0$, respectively. We want to fill it in with a $Z$-scheme morphism $X \to Y$. More generally, suppose $X_0$, and $Y_0$ are flat (of locally finite presentation) over $Z_0$, then when can we find a flat (of locally finite presentation) morphism $X \to Y$?

These problems are solved in great generality by Illusie's cotangent complex, which formally is just the left derived functor (in a suitable derived category) of the Kähler differentials $\Omega_{X/Y}$, thought of as functorial in $X$.[1] We will find that obstructions to deformations are given by a class in $\mathbb{E}xt^2_{\mathcal{O}_{X_0}}(\mathrm{L}_{X_0/Y_0}, J)$, and the set of allowed deformations forms a torsor under $\mathbb{E}xt^1_{\mathcal{O}_{X_0}}(\mathrm{L}_{X_0/Y_0}, J)$, where $J$ is the square-zero ideal of the extension $j : Y_0 \hookrightarrow Y$ (see §5) and $\mathrm{L}_{X_0/Y_0}$ is the relative cotangent complex of $X_0$ over $Y_0$. Here, $\mathbb{E}xt$ is defined in the Appendix; it agrees with the usual Ext if the simplicial modules in question are trivial.

**Remark.** Though we have so far considered deformations of schemes, the theory we will develop will be far more general, instead working with topoi. Although we won't cover this example in this paper, suppose you wanted to deform $f : X \to Y$, where all three, $X$, $Y$, and $f$ can vary. In this case you *must* work in the topos-theoretic setting: There is a certain ringed topos that encodes these data.

Unfortunately, there is not enough space here to cover even the most essential topos theoretic and simplicial algebraic preliminaries. Instead, these most essential pieces have been relegated to an Appendix at the end of the paper. If one does not know a definition or result that is referenced in the body of the paper, then it should (almost always) be found there. After giving the basic notation and definitions to be used in the paper (others will be found in the Appendix), we will define the cotangent complex and give some indication as to why it is the "right" definition.[2] Afterwards, we will indicate the main theorem of deformation theory via the cotangent complex, that $\mathrm{Exal}_A(B, M) \cong \mathbb{E}xt^1(\mathrm{L}_{B/A}, M)$, where the first is the group of $A$-algebra extensions of $B$ by $M$. At this point, we can now tackle deformation-theoretic problems. We will fully characterize obstructions to deformations, isomorphism classes of deformations, and the automorphism group of a given deformation, all of which will be given by various constructions related to the cotangent complex, utilizing in a critical way the preliminary material.

**Remark.** You will notice that the bulk of the paper is discussing algebraic and category-theoretic preliminaries. This is because once these are well-established and sorted, most of the work is done. We need only use the results and definitions discussed previously to tackle deformation theory, and we will find that the proofs are not difficult—most of the work has been relegated to the definitions. This tells us that the theory we are studying is well-developed.

[1] I unfortunately don't have space to include this result.

[2] Whenever a mathematical object is mentioned whose definition is not given in this paper, a definition can be found in [1] and/or in [4].

# 2 Notation and Basic Definitions

Let $\Delta$ denote the simplex category as usual, allowing us to define simplicial and cosimplicial objects in any other category. Denote by $\bar{\Delta}$ the category of *finite* totally ordered sets. Given $I \in \bar{\Delta}$ and a family of functors $(S_i)_{i\in I}$ of a category $\mathsf{A}$ into itself, we can define the composition $\circ_{i\in I} S_i$. We will only need the form of this composition for $I = [0,n]$, in which case $\circ_{i\in I} S_i = S_0 \cdots S_n$, and if $I = \emptyset$, then $\circ_{i\in I} S_i = \mathrm{id}_{\mathsf{A}}$.

Given a $p$-simplicial or $p$-cosimplicial object $X$ of a category $\mathsf{A}$, let $\Delta X$ be the **diagonal (co)simplicial subobject of** $X$, given by $E \mapsto X(E,\ldots,E)$. Given an additive category $\mathsf{A}$, let $C(\mathsf{A})$ be the category of complexes. Define $n$-$C(\mathsf{A})$ to be the category of complexes of $n$-uples of $A$. Let $\mathrm{Simpl}(\mathsf{A})$ be the category of simplicial objects of $\mathsf{A}$. Similarly, let $\mathrm{hSimpl}(\mathsf{A})$ denote the category of simplicial objects of $\mathsf{A}$ up to homotopy. For an $n$-simplicial object $X$ of $\mathsf{A}$, the **complex of chains** of $X$ is denoted $\widetilde{X}$ and given by $\widetilde{X}_{p_1,\ldots,p_n} = X_{p_1,\ldots,p_n}$ for $p_i \geq 0$, where the differential $d_i = \sum_j (-1)^j X(\mathrm{id},\ldots,d-J,\ldots,\mathrm{id})$ ($d_j$ in the $i$'th place). We have a functor of **associated simple complex**

$$\int : n\text{-}C(\mathsf{A}) \to C(\mathsf{A}),$$

defined as $\int L^p = \bigoplus_{\sum p_i = p} L^{p_1,\ldots,p_n}$, with differential given by $\sum_j (-1)^{\sum_{i<j} p_i} d_j$.

Define the **normal complex** $NX$ of $X$ in the usual way,

$$NX_n = \cap_{i\in\mathbb{N}} \ker(d_i : X_n \to X_{n-1}).$$

Define the **Dold-Puppe transform** $KY$ of $Y \in C.(\mathsf{A})$, where $C.$ (resp. $C^{\cdot}$) denotes the full subcategory of $C$ of chains (resp. cochains). Explicitly, we have

$$KY_n = \bigoplus_{p\in\{0,\ldots,n\}} \bigoplus_{f:[0,n]\to[0,p]} Y_{p,f},$$

where $f$ ranges over all *surjective* arrows $[0,n] \to [0,p]$ of $\Delta$ and $Y_{p,f} = Y_p$. Let $D\,\mathrm{Simpl}(\mathsf{A})$ (resp. $D.(\mathsf{A})$) denote $\mathrm{Simpl}(\mathsf{A})$ (resp. $C.(\mathsf{A})$) localized with respect to quasi-isomorphisms.

Let $T$ be a topos and $A$ a ring of $T$ (the prototype is $A = \mathcal{O}$ for $\mathcal{O}$ a structure sheaf of a scheme).[3] Denote by $A_X$ the free $A$-module generated by $X$, i.e. the sheaf associated to the presheaf $U \mapsto A(U)_{X(U)}$. For $M \in A$-$\mathsf{mod}$, denote by $S_A(M)$, or simply $S(M)$, when $A$ is clear from the context, the symmetric algebra of $M$. For $X \in T$, denote

$$A[X] = S_A(A_X),$$

the free $A$-algebra generated by $X$. A **simplicial sheaf** is just a simplicial object of $T$, and it follows by a theorem of Giraud (SGA 4 IV 1.2) that the category $\mathrm{Simpl}(T)$ is a topos. We can then go on to discuss inner homs in this category vs. $T$, but we will not do so (see §2.3.1 of [1]). A **simplicial ring of** $T$ is just a ring of $\mathrm{Simpl}(T)$. Given a simplicial ring $\Lambda$, a **simplicial** $\Lambda$**-module** (resp. a **simplicial** $\Lambda$**-algebra**) is a module (resp. an algebra) over $\Lambda$ viewed as a **trivial simplicial object**, i.e. one defined by a constant functor $\Delta^{\mathrm{op}} \to \mathsf{C}$, where $\mathsf{C}$ is the category of interest. As mentioned in §1, we denote by $\mathbb{E}xt$ the usual Ext in the category of simplicial modules.

[3] I'm going to assume almost everything from topos theory as prior knowledge; however, we don't need this generality to understand the bulk of the paper. Namely, you can think of a topos as just the category of sheaves on a site. If this is uncomfortable, just think of the category of sheaves on a topological space.

Given a simplicial ring $A$, denote by $A$-mod the category of $A$-modules and by $A$-alg that of simplicial $A$-algebras. Finally, given a fixed $A$-algebra $B$, we write $A\text{-alg}/B$ for the category of $A$-algebras "above" $B$, i.e. equipped with a morphism to $B$ that fits in a suitable commutative triangle. We say a morphism of $A$-mod (resp. $A$-alg, resp. $A\text{-alg}/B$) is a **quasi-isomorphism** if the morphism of the underlying simplicial sheaves of sets is one. We denote by $D.(A)$ (resp. $D(A\text{-alg})$, resp. $D(A\text{-alg}/B)$) the corresponding localized categories with respect to quasi-isomorphisms.

# 3 The Cotangent Complex

First, we recall/make some definitions. Here, $T$ is a topos, $A$ a ring of $T$, $B$ an $A$-algebra, and $M$ a $B$-module. Denote by

$$B \oplus M = S_B(M)/\oplus_{i\geq 2} S^i_B(M)$$

the **$B$-algebra of dual numbers on $M$**. There is then a canonical functorial isomorphism

$$\begin{gathered}\mathrm{Der}_A(B,M) \xrightarrow{\cong} \mathrm{Hom}_{A\text{-alg}/B}(B, B\oplus M)\\ d \mapsto (x \mapsto x + dx).\end{gathered}$$

There is an exact sequence

$$\begin{gathered}0 \to I \to B\otimes_A B \to B \to 0\\ x\otimes y \mapsto xy\end{gathered}$$

split by the two ring morphisms

$$\begin{gathered}j_1, j_2 : B \to B\otimes_A B\\ j_1(x) = x\otimes 1\\ j_2(x) = 1\otimes x.\end{gathered}$$

Define, for $n \in \mathbb{N}$,

$$P^n_{B/A} = B\otimes_A B/I^{n+1},$$

the **ring of principal parts of order** $n$ **of** $B$ **over** $A$. The structure of $B$-algebra on $P^n_{B/A}$ defined by $j_1$ (resp. by $j_2$) will be called the **left** (resp. **right**) **structure**, and when we regard $P^n_{B/A}$ as a $B$-algebra without specification, we mean the left structure. Using $P^1_{B/A}$ we define the module of Kähler differentials of $B$ over $A$ as usual, which is just $\Omega^1_{B/A} = I/I^2$. Recall that the exterior differential $d_{B/A}$ gives us a functorial isomorphism

$$\begin{gathered}\mathrm{Hom}_B(\Omega^1_{B/A}, M) \xrightarrow{\cong} \mathrm{Der}_A(B,M)\\ u \mapsto u d_{B/A}.\end{gathered}$$

**Remark.** In the sequel we will begin to speak of morphisms of ringed topoi. Although formally we really do mean this, you can just think of these as morphisms induced by scheme morphisms.

Let $f : X \to Y$ be a morphism of ringed topoi. Define

$$P^n_{X/Y} := P^n_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}$$
$$\Omega^1_{X/Y} := \Omega^1_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}.$$

As before, we say that $P^n_{X/Y}$ is the ring of principal parts of order $n$ of $X$ over $Y$, the topos $X$ ringed by $P^n_{X/Y}$ is called the $n$'th infinitesimal $Y$-neighborhood of $X$ in the diagonal. We say that $\Omega^1_{X/Y}$ is the module of Kähler differentials of $X$ over $Y$. To a commutative diagram of ringed topoi

$$\begin{array}{ccc} X & \xleftarrow[u]{} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow[v]{} & Y' \end{array}, \tag{4}$$

corresponds a morphism $\Omega^1_{X/Y} \to \Omega^1_{X'/Y'}$ of modules above $u$, i.e. a morphism

$$u^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y'} \tag{5}$$

defined as follows. Let $A = u^{-1}f^{-1}\mathcal{O}_Y \cong f'^{-1}v^{-1}\mathcal{O}_Y$, $B = u^{-1}\mathcal{O}_X$, $A' = f'^{-1}\mathcal{O}_{X'}$, $B' = \mathcal{O}_{X'}$, from which we obtain a commutative square of rings of $X'$

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}. \tag{6}$$

The morphism (5) is thus the map

$$\Omega^1_{B/A} \otimes_B B' \to \Omega^1_{B'/A'}$$

defined by this square and the identification of $\Omega^1_{B/A}$ and $u^{-1}\Omega^1_{X/Y}$ by the definition of $\Omega^1_{X/Y}$. If the square is cocartesian, (5) is thus an isomorphism by definition. If we have a morphisms of rings of $T$, $A \to B \to C$, these give the usual exact sequence of $C$-modules for $\Omega^1$. Similarly, if we instead have functorial morphisms defined by morphisms of ringed topoi, $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have the exact sequence

$$f^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0.$$

Let $A \to B$ be a morphism of rings of $T$. We define the **cotangent complex of $B$ over $A$**, denoted $\mathrm{L}_{B/A}$, the simplicial $B$-module

$$\mathrm{L}_{B/A} := \Omega^1_{P/A} \otimes_P B,$$

where $P := P_A(B)$ is the standard simplicial resolution of $B$. By functoriality of the standard resolution and the module of Kähler differentials, the cotangent complex is also functorial in many ways. For example, it depends functorially on the morphism $A \to B$: To a commutative square (6) is associated a simplicial $B$-module morphism $\mathrm{L}_{B/A} \to \mathrm{L}_{B'/A'}$. It also commutes with filtered colimits. It commutes with pullback $f^{-1}\mathrm{L}_{B/A} \cong \mathrm{L}_{f^{-1}B/f^{-1}A}$ and the sheafification functor. See §II1.2.3 of [1] for details on these. The augmentation $P_A(B) \to B$ defines an augmentation

$$\mathrm{L}_{B/A} \to \Omega^1_{B/A}. \tag{7}$$

**Proposition 1.** The morphism

$$\mathrm{H}_0(\mathrm{L}_{B/A}) \to \Omega^1_{B/A}$$

defined by the augmentation (7) is an isomorphism.

*Proof.* Denote $P_A(B) = P$ again. The fact that $P \to B$ is a quasi-isomorphism implies in particular that the $A$-algebra $B$ is the cokernel of the double arrow $P_1 \rightrightarrows P_0$. The claim hence follows from the fact that $\Omega^1$ commutes with colimits. □

**Corollary 2.** Let $M$ be a $B$-module. There exists a canonical functorial isomorphism

$$\mathbb{E}xt^0_B(\mathrm{L}_{B/A}, M) \xrightarrow{\cong} \mathrm{Der}_A(B, M).$$

Let now $A \to B$ be a morphism of simplicial rings of $T$. Then $\mathrm{L}_{B/A}$ is a bisimplicial $B$-module, where we regard $B$ as a bisimplicial ring that is trivial in the vertical direction. We set

$$\mathrm{L}^{\Delta}_{B/A} := \Delta \mathrm{L}_{B/A} = \Omega^1_{P/A} \otimes_P B.$$

This is a flat $B$-module, which, when $A \to B$ is defined by a morphism of rings $A_0 \to B_0$ of $T$, coincides with the cotangent complex $\mathrm{L}_{B_0/A_0}$ introduced earlier. The augmentation (7) regarded as a morphism of bisimplicial $B$-modules induces, by restriction to the diagonal, a morphism of $B$-modules

$$\mathrm{L}^{\Delta}_{B/A} \to \Omega^1_{B/A}. \tag{8}$$

We have

**Proposition 3.** Suppose that for each $n \in \mathbb{N}$, there exists a flat $A_n$-module $E_n$ such that $B_n \cong S_{A_n}(E_n)$ (which is the case for example if $B$ is free over $A$ term-by-term). Then, the augmentation (8) is a quasi-isomorphism.

*Proof.* Omitted; see II.1.2.5.3 and II.1.2.4.4 of [1]. □

Let $f : X \to Y$ be a morphism of ringed topoi. We call the **cotangent complex of $X$ over $Y$** the simplicial $\mathcal{O}_X$-module $\mathrm{L}_{X/Y}$ defined by

$$\mathrm{L}_{X/Y} := \mathrm{L}_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}.$$

It satisfies a similar functoriality condition on $f$ as we have already seen, giving rise to a morphism $u *$ $\mathrm{L}_{X/Y} \to \mathrm{L}_{X'/Y'}$ associated to a commutative diagram (4).

Let $T$ be a topos, $A \to B \to C$ morphisms of simplicial rings of $T$. We thus have a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & P & \longrightarrow & Q \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\ A & \longrightarrow & B & \longrightarrow & C \end{array},$$

where $P = P^{\Delta}_A(B) := \Delta P_A(B)$ and $Q = P^{\Delta}_P(C)$. Since $Q$ is free term-by-term over $P$, the canonical sequence

$$0 \to \Omega^1_{P/A} \otimes_P Q \to \Omega^1_{Q/A} \to \Omega^1_{Q/P} \to 0 \tag{9}$$

is exact. Since $\Omega^1_{Q/P}$ is flat over $Q$, extension by scalars in $C$ of the sequence above is likewise exact. Define

$$\mathrm{L}^{\Delta}_{C/B/A} = (9) \otimes_Q C;$$

this sequence is exact. It is called the **exact sequence** (or **true triangle**) **of transitivity relative to** $A \to B \to C$, and it depends functorially on $A \to B \to C$.

**Proposition (Distinguished or Fundamental Triangle of the Cotangent Complex).** The distinguished triangle of $D.(C)$ associated to $L^{\Delta}_{C/B/A}$ by the functor $\boldsymbol{\chi}$ is, up to a canonical isomorphism,

$$\begin{array}{ccc} & \mathrm{L}^{\Delta}_{C/B} & \\ \swarrow & & \nwarrow \\ \mathrm{L}^{\Delta}_{B/A} \otimes_B C & \longrightarrow & \mathrm{L}^{\Delta}_{C/A} \end{array} , \qquad (10)$$

where the left arrow is a map of degree 1 and the degree 0 arrows are the images in $D.(C)$ of the canonical ones.

We omit the proof for brevity, see II.2.1.2 of [1].

# 4 The Fundamental Theorem

Let $T$ be a topos. By **extension of algebras** of $T$ we mean a commutative diagram of rings of $T$

$$X = \begin{array}{ccc} E & \xrightarrow{p} & B \\ & \nwarrow & \uparrow \\ & & A \end{array} ,$$

where $p$ is surjective and the kernel of $p$ is a square-zero ideal $I$. We say that $X$ is an $A$-**extension of** $B$ **by the** $B$-**module** $I$. The algebra extensions of $T$ form a category, denoted $\underline{\mathsf{Exal}}$, where morphisms of algebra extensions are given by commutative diagrams in the obvious way (start with a map $A' \to A \dots$). By associating to each extension $X$ the pair formed by an underlying morphism $A \to B$ and the $B$-module $I = p^{-1}(0)$, we can define a functor

$$\pi : \underline{\mathsf{Exal}} \to \mathsf{Algmod},$$

where Algmod is the category of pairs $(A \to B, I)$ where $A \to B$ is a morphism of rings and $I$ is a $B$-module. A morphism between such pairs is defined by a commutative square in the natural way. Fixing a ring $A$, we denote by $\underline{\mathsf{Exal}}_A$ the subcategory of $\underline{\mathsf{Exal}}$ formed by $A$-extensions. We denote by $\underline{\mathsf{Exal}}_A(B,-)$ the subcategory of $A$-extensions of $B$ by a variable $B$-module, and finally we denote by $\underline{\mathsf{Exal}}_A(B,I)$ the subcategory of $A$-extensions of $B$ by $I$.

Think of $X$ as a short exact sequence $I \to E \to B$. Given a map $f : E \to E'$ (resp. $g : G' \to G$) of $A$-modules, denote by $f * X$ the pushout (resp. by $X * g$ the pullback) extension: $0 \to E' \to E' \overset{E}{\oplus} F \to G \to 0$

(resp. $0 \to E \to F \times_G G' \to G' \to 0$), where $E' \overset{E}{\oplus} F$ denotes the obvious pushout. Now, given a morphism of $A$-algebras $u : C \to B$, we have $X * u$. There is a canonical morphism $X * u \to X$ in $\underline{\mathsf{Exal}}_A$ whose image under $\pi$ is $(u, \mathrm{id}_I)$. Moreover, given $u' : C' \to C$ another morphism of $A$-algebras, we have a canonical isomorphism

$$X * (uu') \xrightarrow{\cong} (X * u) * u'$$

in $\underline{\mathsf{Exal}}_A(C', I)$, satisfying the usual cocycle condition. Similarly, given $v : I \to J$ a morphism of $B$-modules, we can define $v * X$ as before, we have a canonical morphism $X \to v * X$ whose image under $\pi$ is $(\mathrm{id}_B, V)$, and we have a canonical isomorphism

$$(v'v) * X \xrightarrow{\cong} v' * (v * X)$$

satisfying the usual cocycle condition.

Let $C$ be an $A$-algebra. We say that $C$ **satisfies condition** (L) if for any $A$-extension $X$

$$0 \to I \to B \to C \to 0,$$

the sequence

$$0 \to I \to \Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to 0$$

is exact on the left; denote this sequence by $\mathrm{diff}(X)$. We have thus obtained a functor $X \mapsto \mathrm{diff}(X)$ of $\underline{\mathsf{Exal}}_A(C, -)$ compatible with the natural projections onto $C$-mod and giving rise to a group homomomorphism

$$\mathrm{diff} : \mathrm{Exal}_A(C, I) \to \mathrm{Ext}^1_C(\Omega^1_{C/A}, I),$$

where we denote by $\mathrm{Exal}$ the Picard group underlying $\underline{\mathsf{Exal}}$.

Now, let

$$Y = (0 \to I \xrightarrow{i} J \xrightarrow{f} \Omega^1_{C/A} \to 0)$$

be an exact sequence of $C$-modules. From it, we can obtain an $A$-extension of $P^1_{C/A} \cong C \oplus \Omega^1_{C/A}$ by $I$

$$Y' = (0 \to I \xrightarrow{\begin{bmatrix} 0 \\ i \end{bmatrix}} C \oplus J \xrightarrow{\mathrm{id} \oplus f} C \oplus \Omega^1_{C/A} \to 0).$$

Next, define

$$\mathrm{alg}(Y) := Y' * j_2,$$

where $j_2 : C \to P^1_{C/A}$ is the homomomorphism defining the right structure on $P^1_{C/A}$, i.e. $j_2(x) = x + d_{C/A}x$. We then have the following, which we state without proof (see II.1.1.9 of [1]).

**Proposition 4.** If $C$ is an $A$-algebra verifying condition (L), then $\mathrm{diff}$ and $\mathrm{alg}$ are inverses to one another.

Let $f : X \to Y$ be a morphism of ringed topoi. An **infinitesimal $Y$-neighborhood of $X$ of order 1** (or **$Y$-extension**) **of** $X$ will mean a commutative diagram of ringed topoi

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & X' \\ \big\downarrow{\scriptstyle f} & \swarrow & \\ Y & & \end{array},$$

where $i$ is an equivalence of the underlying topoi and $i^{-1}\mathcal{O}_{X'} \to \mathcal{O}_X$ is surjective with a square-zero kernel. We are now ready to state

**Theorem 5.** Let $T$ be a topos, $A \to B$ a morphism of rings of $T$, and $M$ a $B$-module. There is a functorial isomorphism

$$\mathrm{Exal}_A(B, M) \xrightarrow{\cong} \mathbb{E}xt^1_B(\mathrm{L}_{B/A}, M).$$

The proof will be given after some preliminaries. Denote by $P = P_A(B)$ as usual. It satisfies condition (L) since it is an $A$-algebra of the simplicial topos $\mathrm{Simpl}(T)$ and it is free term-by-term. Let $X$ be an $A$-extension of $B$ by $M$. We thus obtain an $A$-extension $X * p$ of $P$ by $M$ (here $p$ is the augmentation $P \to B$), hence also an extension of $P$-modules $\mathrm{diff}(X * p)$ of $\Omega^1_{P/A}$ by $M$. Finally, we have an element

$$\alpha(X) = \chi\,\mathrm{diff}(X * p) \in \mathbb{E}xt^1_P(\Omega^1_{P/A}, M).$$

Clearly $X \mapsto \alpha(X)$ is a group homomorphism which is functorial in $M$. It is also functorial in the map $A \to B$, see §II.1.2.2 [1]. Moreover, using the isomorphism (13), we have

$$\mathbb{E}xt^1_P(\Omega^1_{P/A}, M) \xrightarrow{\cong} \mathbb{E}xt^1_B(\mathrm{L}_{B/A}, M).$$

*Proof Sketch of Theorem.* We claim that $\alpha$ composed with the above isomorphism gives us the desired one. We will proceed by constructing an inverse to $\alpha$. Let $y \in \mathbb{E}xt^1_P(\Omega^1_{P/A}, M)$. Using I.3.2.3.8 [1], there exists a quasi-isomorphism of $P$-modules $s : M \to N$ and an extension $Y$ of $\Omega^1_{P/A}$ by $N$ such that $y = \sigma s^{-1}\chi(Y)$, giving us an $A$-extension

$$\mathrm{alg}(Y) = (0 \to N \to E \to P \to 0).$$

Since $P$ is acyclic in positive degrees, we can apply $\mathrm{H}_0$ to deduce an $A$-extension of $\mathrm{H}_0(P)$ by $\mathrm{H}_0(N)$, finally obtaining an $A$-extension of $B$ by $M$:

$$(\mathrm{H}_0 s)^{-1} * \mathrm{H}_0\,\mathrm{alg}(Y) * (\mathrm{H}_0 p)^{-1} = (0 \to M \to F \cong \mathrm{H}_0 E \to B \to 0).$$

It then takes some work to prove that this is independent of choice of $(s, Y)$, which will then allow us to define $\beta(y) = (\mathrm{H}_0 s)^{-1} * \mathrm{H}_0\,\mathrm{alg}(Y) * (\mathrm{H}_0 p)^{-1}$. Skipping this work, we have indeed,

$$\begin{aligned} \beta\alpha(X) =& \beta(\chi\,\mathrm{diff}(X * p)) \\ =& \mathrm{H}_0(\mathrm{alg}\,\mathrm{diff}(X * p)) * \mathrm{H}_0 p^{-1} \\ =& \mathrm{H}_0(X * p) * H_0 p^{-1} \\ =& X, \end{aligned}$$

so $\beta\alpha = \mathrm{id}$. Conversely, suppose $y \in \mathbb{E}xt^1_P(\Omega^1_{P/A}, M)$. Then, by definition

$$\begin{aligned}\alpha\beta(y) =& \chi\,\mathrm{diff}(\mathrm{H}_0 s^{-1} * \mathrm{H}_0\,\mathrm{alg}(Y) * \mathrm{H}_0\, p^{-1} * p)\\ =& \sigma \mathrm{H}_0 s^{-1} \chi\,\mathrm{diff}(\mathrm{H}_0\,\mathrm{alg}(Y) * \mathrm{H}_0\, p^{-1} * p) \qquad (\because \chi(f * X) = (\sigma f)\chi(X)).\end{aligned}$$

The morphism of $A$-extensions

$$\begin{array}{lccccccccc} \mathrm{alg}(Y) & 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & P & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle v} & & \downarrow & & \downarrow{\scriptstyle (\mathrm{H}_0 p)^{-1} p} & & \\ \mathrm{H}_0\,\mathrm{alg}(Y) & 0 & \longrightarrow & \mathrm{H}_0 N & \longrightarrow & \mathrm{H}_0 E & \longrightarrow & \mathrm{H}_0 P & \longrightarrow & 0 \end{array}$$

defined by projection onto $\mathrm{H}_0$ gives the relation

$$\mathrm{H}_0\,\mathrm{alg}(Y) * \mathrm{H}_0\, p^{-1} * p = v * \mathrm{alg}(Y).$$

Thus,

$$\begin{aligned}\alpha\beta(y) =& \sigma((\mathrm{H}_0 s^{-1})v)\chi\,\mathrm{diff}\,\mathrm{alg}(Y)\\ =& \sigma((\mathrm{H}_0 s^{-1})v)\chi(Y).\end{aligned}$$

However, we have $vs = \mathrm{H}_0 s$, so $(\mathrm{H}_0 s^{-1})v = s^{-1}$ in $D.(P)$. Hence,

$$\alpha\beta(y) = \sigma s^{-1}\chi(Y) = y.$$

□

This theorem implies that $A$-algebra extensions of $B$ by $M$ are completely controlled by $\mathrm{L}_{B/A}$. Finally, we have

**Lemma (EGA $0_{\mathrm{IV}}$ 18.3.8).** Let $A \to B$ be a surjective morphism of rings with kernel $I$. Denote by $U$ the $A$-extension $I/I^2 \to A/I^2 \to b$. Then $f \mapsto f * U$ defines a functorial isomorphism

$$\mathrm{Hom}_B(I/I^2, M) \xrightarrow{\cong} \mathrm{Exal}_A(B, M)$$

for $M \in B$-mod.

**Corollary 6.** Under the hypotheses of the lemma above, we have $\mathrm{H}_0(\mathrm{L}_{B/A}) = 0$ and a functorial canonical isomorphism $\mathrm{H}_1(\mathrm{L}_{B/A}) \cong I/I^2$.

*Proof.* Since the product $B \otimes_A B \to B$ is an isomorphism, we have $\Omega^1_{B/A} = 0$, so $\mathrm{H}_0(\mathrm{L}_{B/A}) = 0$. Hence, the canonical projection $\mathrm{L}_{B/A} \to \mathrm{H}_1(\mathrm{L}_{B/A})[1]$ defines a functorial isomorphism

$$\mathrm{Hom}_B(\mathrm{H}_1(\mathrm{L}_{B/A}), M) \xrightarrow{\cong} \mathbb{E}xt^1_B(\mathrm{L}_{B/A}, M)$$

for $M \in B$-mod. Using Theorem 5 and the lemma above, we have the isomorphism

$$\mathrm{Hom}_B(I/I^2, M) \xrightarrow{\cong} \mathrm{Hom}_B(\mathrm{H}_1(\mathrm{L}_{B/A}, M).$$

□

# 5 Deformation Theory

Fix a topos $T$ and a ring $A$ of $T$. Recall that we have a functor

$$\pi : \underline{\mathsf{Exal}}_A \to A\text{-}\mathsf{algmod},$$

where $A$-algmod is the subcategory of Algmod consisting of pairs $(B, M)$ where $B$ is an $A$-algebra and $M$ a $B$-module (a morphism is then pair consisting of a morphism of $A$-algebras and a morphism of $B$-modules). $\pi$ associates to each $A$-extension $0 \to I \to B \to B_0$ the pair $(B_0, I)$ and to each morphism of $A$-extensions $\bar{B} \xrightarrow{\bar{f}} \bar{C}$ the pair $\pi(\bar{f}) = (f_0, u)$, where $u : I \to J$, $f : B \to C$, $f_0 : B_0 \to C_0$. Now, fix an $A$-extension $\bar{B}$ and a morphism of $A$-algmod $(f_0 : B_0 \to C_0, u : I \to J)$. Consider the following

**Problem.** Find a morphism of $A$-extensions $\bar{f} : \bar{B} \to \bar{C}$ such that $\pi(\bar{f}) = (f_0, u)$.

We can reformulate this problem as follows. Find a $B$-extension $\bar{C}$ of $C_0$ by $J$ such that the image of the compositie map

$$\mathrm{Exal}_B(C_0, J) \to \mathrm{Exal}_B(B_0, J) \xrightarrow{\cong} \mathrm{Hom}_{B_0}(I, J)$$

is the given morphism $u$, the first arrow is the canonical arrow defined by $f_0$, and the second the canonical isomorphism of (EGA $0_{\mathrm{IV}}$ 18.3.8). We have a commutative square

$$\begin{array}{ccc} \mathrm{Exal}_B(C_0, J) & \longrightarrow & \mathrm{Hom}_{B_0}(I, J) \\ \downarrow{\scriptstyle \cong \text{ Theorem 5}} & & \downarrow{\scriptstyle \cong \text{Cor. 8}} \\ \mathbb{E}xt^1_{C_0}(L_{C_0/B}, J) & \longrightarrow & \mathbb{E}xt^1_{C_0}(\mathrm{L}_{B_0/B} \otimes_{B_0} C_0, J) \end{array},$$

where the lower horizontal arrow is defined by the canonical morphism $\mathrm{L}_{B_0/B} \otimes_{B_0} C_0 \to \mathrm{L}_{C_0/B}$. It follows that the problem is solved by the long exact sequence of $\mathbb{E}xt^i_{C_0}(-, J)$ defined by the fundamental transitivity triangle of $\mathrm{L}_{C_0/B_0/B}$:

$$0 \to \mathrm{Der}_{B_0}(C_0, J) \to \mathrm{Der}_B(C_0, J) \to 0 \to \mathrm{Exal}_{B_0}(C_0, J) \to$$
$$\to \mathrm{Exal}_B(C_0, J) \to \mathrm{Hom}_{B_0}(I, J) \xrightarrow{\partial} \mathbb{E}xt^2_{C_0}(\mathrm{L}_{C_0/B_0}, J) \to \cdots.$$

We thus have

**Theorem 7.** There exists an obstruction $\omega(\bar{B}, f_0, u) = \partial u \in \mathbb{E}xt^2_{C_0}(\mathrm{L}_{C_0/B_0}, J)$ whose vanishing is necessary and sufficient for the problem above admitting a solution. Moreover, when it does vanish, the set of isomorphism classes of solutions to the problem is a torsor under the group $\mathrm{Exal}_{B_0}(C_0, J)$. The group of automorphisms of a solution is canonically identified with $\mathrm{Der}_{B_0}(C_0, J)$.

**Remark.** This is the simplest situation; there are various improvements we could make to this theorem. For example, suppose we instead want a flat deformation of $C_0$ over $B$, i.e. a $B$-extension $\bar{C}$ such that $C$ is flat over $B$ and $C \otimes_B B_0 \to C_0$ is an isomorphism. Notice that the class $\omega(\bar{B}, f_0, u)$ depends functorially on $u$. More precisely, if we denote by $\omega(\bar{B}, f_0) \in \mathbb{E}xt^2_{C_0}(\mathrm{L}_{C_0/B_0}, I \otimes_{B_0} C_0)$ the obstruction class corresponding to the adjunction morphism $I \to I \otimes_{B_0} C_0$, so that for a morphism $u : I \to J$ corresponding by adjunction to $v : I \otimes_{B_0} C_0 \to J$, we have

$$\omega(\bar{B}, f_0, u) = v\omega(\bar{B}, f_0).$$

We then immediately obtain

**Corollary 8.** The class $\omega(\bar{B}, f_0)$ is the obstruction fo a flat deformation of $C_0$ over $B$. When it vanishes, the set of isomorphism classes of flat deformations is a torsor under $\mathrm{Exal}_{B_0}(C_0, I \otimes_{B_0} C_0)$, and the group of automorphisms of a flat deformation is canonically identified with $\mathrm{Der}_{B_0}(C_0, I \otimes_{B_0} C_0)$.

We can now state the solution to Problems 0 and 1 from the introduction. Note that in addition to the statements there, we also have on the topos $|X_0|$, which denotes the underlying topos of the ringed topos $X_0$, a morphism of $p_0^{-1}(\mathcal{O}_S)$-extensions:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X_0} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & f_0^{-1}J & \longrightarrow & f_0^{-1}\mathcal{O}_Y & \longrightarrow & f_0^{-1}\mathcal{O}_{Y_0} & \longrightarrow & 0 \end{array},$$

where we identify $|X|$ with $|X_0|$ and $|Y|$ with $|Y_0|$ via $|i|$ and $|j|$. In particular, we have a morphism of $\mathcal{O}_{X_0}$-modules, $v : f_0^* J \to I$. And, if $f_0$ is flat, $f$ is flat if and only if $v$ is an isomorphism (see Lemma III.2.1.1.1 of [1]; this is just commutative algebra).

**Theorem 9.** There exists an obstruction $\omega(f_0, j, v) \in \mathbb{E}xt^2_{\mathcal{O}_{X_0}}(\mathrm{L}_{X_0/Y_0}, I)$ whose vanishing is necessary and sufficient for the existence of a diagram (1) inducing the diagram (2) and the morphism $v$ given above. When this class does vanish, the set of isomorphism classes of solutions (2) is a torsor under the group $\mathbb{E}xt^1_{\mathcal{O}_{X_0}}(\mathrm{L}_{X_0/Y_0}, I)$, and the automorphism group of a solution is canonically identified with $\mathbb{E}xt^0_{\mathcal{O}_{X_0}}(\mathrm{L}_{X_0/Y_0}, I)$. Moreover, we have $\omega(f_0, j, v) = v\omega(f_0, j)$ where we set

$$\omega(f_0, j) = \omega(f_0, j, \mathrm{id}(f_0^* J)) \in \mathbb{E}xt^2_{\mathcal{O}_{X_0}}(\mathrm{L}_{X_0/Y_0}, f_0^* J).$$

When $f_0$ is flat, $\omega(f_0, j)$ is an obstruction to the existence of a flat deformation of $X_0$ over $Y$. If it vanishes, the set of isomorphism classes is a torsor under $\mathbb{E}xt^1_{\mathcal{O}_{X_0}}(\mathrm{L}_{X_0/Y_0}, f_0^* J)$ and the group of automorphisms of a given flat deformation is canonically identified with $\mathbb{E}xt^0_{\mathcal{O}_{X_0}}(\mathrm{L}_{X_0/Y_0}, f_0^* J)$.

**Remark 1.** This theorem accounts for the entire theory of scheme deformations. Indeed, suppose that the topoi $X_0, Y_0, S$ are locally ringed and the morphisms $f_0, q_0$ are admissible in the sense of M. Hakim's thesis (this is the case if for example the topoi are those associated to schemes). It follows immediately by definition that $X \to Y \to S$ are locally ringed and $i, j$ are admissible, so our problem is still solved by Theorem 9.

**Remark 2.** Suppose that $X_0 \to Y_0 \to S$ really are scheme morphisms (i.e. morphisms of ringed Zariski topoi defined by scheme morphisms) and that $I$ (resp. $J$) is a quasi-coherent $\mathcal{O}_{X_0}$-(resp. $\mathcal{O}_{Y_0}$-)module. Then it follows from (EGA I 2e édition 5.1.9) that $X$ and $Y$ are schemes and the morphisms $X \to Y \to S$, $i, j$ are scheme morphisms. Hence, our problem is again solved by Theorem 9.

**Remark 3.** Finally, suppose that $X_0$ is a flat relative scheme locally of finite presentation over $Y_0$ and that $v$ is an isomorphism. The definition is a bit technical, but such a relative scheme is defined by the following.

i) a family $\{U_i\}_{i \in I}$ of objects of $S$ which cover the final object of $S$

ii) for all $i \in I$, a $\Gamma(U_i, \mathcal{O}_S)$-scheme $X_i$

iii) for each pair $(i, j) \in I \times I$, restriction isomorphisms of $X_i$ and $X_j$ on a sufficiently fine refinement of $U_i \times U_j$ that verify a condition of transitivity over a sufficiently fine refinement of $U_i \times U_j \times U_k$.

In effect, a flat relative scheme is just a family of schemes. We need the locally of finite presentation condition to ensure that the definitions of flatness for such general objects agree (there are different ones, but they coincide if we include this condition, see [3] for details). It then follows immediately that $X$ is a flat relative scheme locally of finite presentation over $Y$, so this problem is again solved by Theorem 9.

We now want to consider morphisms of schemes (or more generally ringed topoi), i.e. Problem 2 from the introduction, but unfortunately there is not enough space at this point. If the reader has been following along, it should now be straightforward for him or her to read §III.2.2 of [1], where one can find a theorem characterizing deformations of morphisms in an analogous manner to Theorem 9.

We're now ready for a an example.

**Example Deformations of Curves.** Let $Y$ be a locally noetherian scheme. We define a **curve over** $Y$ to be a morphism $f : X \to Y$ which is flat, separated, of finite type, and with relative dimension 1. We take $f$ to be proper. Let $A$ be a complete local noetherian ring, with residue field $k$. Let $S = \operatorname{Spec} A$, $s = \operatorname{Spec} k$, and let $X_0$ be a projective and smooth scheme over $s$ satisfying $H^2(X_0, T_{X_0/s}) = 0$. We then claim that there exists a proper and smooth formal scheme $X$ over $\hat{S}$ lifting $X_0$.

*Proof.* Let $\hat{S} = \operatorname{colim} S_n$, where $S_n = \operatorname{Spec} A/\mathfrak{m}^{n+1}$, $\mathfrak{m}$ denoting the maximal ideal of $A$. We claim that $X = \operatorname{colim} X_n$ exists. Assume $X_m$ smooth over $S_m$ has been constructed for $m \le n$ such that $X_m = S_m \times_{S_n} X_n$, and let $i_n : S_n \to S_{n+1}$ be the inclusion. By Proposition III.3.1.2 in [1], $f : X \to Y$ smooth implies that the cotangent complex has perfect amplitude contained in $[0, 0]$, $\Omega^1_{X/Y}$ is locally free of finite type, and the natural augmentation $L_{X/Y} \to \Omega^1_{X/Y}$ is a quasi-isomorphism. In particular, this implies that when we pass to the derived category it is an isomorphism. Then, by Theorem 9, the obstruction to the existence of a smooth lifting $X_{n+1}$ of $X_n$ over $S_{n+1}$ is in fact contained in $\mathrm{H}^2(X_0, T_{X_0/s} \otimes \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}) = \mathrm{H}^2(X_0, T_{X_0/s}) \otimes \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} = 0$ by the assumption; indeed, this follows since $\mathbb{E}xt$ is taken as a colimit in the derived category; hence, we are literally taking $\mathrm{H}^2$. The square zero ideal in question is indeed $\mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$ since it is the kernel of the map $A/\mathfrak{m}^{n+2} \to A/\mathfrak{m}^{n+1}$. □

# References


[1] Luc Illusie, *Compléxe cotangent et déformations I*, Lec. Notes Math. 239, Springer 1971, xv+355 pp.; II, LNM 283, Springer 1972. vii+304 xv+355 pp.

[2] Luc Illusie, Grothendieck's existence theorem in formal geometry (179–233) MR2223409

[3] Monique Hakim, *Topos annelés et schémas relatifs*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 64 Springer (1972)

[4] The Stacks Project, https://stacks.math.columbia.edu/

[5] Martin Olsson, "Tangent Spaces and Obstruction Theories," MSRI Lecture Notes, https://math.berkeley.edu/ molsson/MSRISummer07.pdf.

# 6 Appendix

This material covers the bare minimum of that in Chapters I and II of [1] that we will need. Necessarily, much material will be omitted; you can find further details in [1] or [Stacks]. We begin with

**Theorem (Eilenberg-Zilber).** There are functorial homomorphisms

$$\int \widetilde{X} \to \widetilde{\Delta X}$$
$$\widetilde{\Delta X} \to \int \widetilde{X},$$

inducing the identity in degree 0, and inverse to each other up to a functorial homotopy.

Moreover, these are explicitly given by the "shuffle-map" and the "Alexander-Whitney morphism," though we won't need these explicit descriptions. We also have

**Theorem (Dold-Puppe).** The functors $N : \mathrm{Simpl}(\mathsf{A}) \to C.(\mathsf{A})$ and $K : C.(\mathsf{A}) \to \mathrm{Simpl}(A)$ are quasi-inverse.

This theorem has the following corrolaries.

i) $N$ and $K$ are exact.

ii) There is a functorial canonical isomorphism $\widetilde{X} \cong NX \oplus DX$, where $DX$ is the **degenerate subcomplex** of $X$, given by

$$DX_n = \sum \mathrm{im}(s_i : X_{n-1} \to X_n).$$

We now need one more construction, known as the "standard simplicial resolution." This will allow us to obtain free or flat resolutions of various objects and will be a key component in the definition of the cotangent complex. Let $T : \mathsf{A} \to \mathsf{B}$, $U : \mathsf{B} \to \mathsf{A}$ be a pair of adjoint functors, with adjunction morphisms $a$ and $b$, respectively. We can define an augmented simplicial object $(T, U).$ in $\mathrm{End}(\mathsf{B})$ called the **standard simplicial resolution defined by** $(T, U)$ as follows.

$$\begin{aligned}
(T,U)_n &= (TU)^{[0,n]} \\
(T,U)_{-1} &= (T,U)(\emptyset) = \mathrm{id}_B \\
d_i^n &= (TU)^{[0,i-1]} b (TU)^{[i+1,n]} \\
s_i^n &= (TU)^{[0,i-1]} TaU (TU)^{[i+1,n]},
\end{aligned}$$

where the augmentation is given by $b : TU \to \mathrm{id}_B$. Dually, we can define an object $(U,T)^{\cdot}$ in $\mathrm{End}(\mathsf{A})$, called the **standard cosimplicial resolution defined by** $(T, U)$:

$$\begin{aligned}
(U,T)^n &= (UT)^{[0,n]} \\
(U,T)^{-1} &= (U,T)(\emptyset) = \mathrm{id}_A \\
d_n^i &= (UT)^{[0,i-1]} a (UT)^{[i,n]} \\
s_n^i &= (UT)^{[0,i-1]} UbT (UT)^{i+2,n]},
\end{aligned}$$

with augmentation given by $a : \mathrm{id}_A \to UT$.

Armed with this construction, let now $T$ be a topos and $A$ a ring of $T$. Note that the free $A$-module functor $A_-$ is the left adjoint of the forgetful functor $A\text{-mod} \to T$. Using this pair of adjoint functors $T = A_-$, $U = \mathrm{forget}$, denote $F_A(M) \to M$ (or $F(M)$ if $A$ is clear from the context) the **free standard resoution of** $M$, defined by the standard simplicial resolution associated to $(T, U)$. Similarly, $A[-]$ is left adjoint to the forgetful functor $A\text{-alg} \to A\text{-mod}$. For $B \in A\text{-alg}$, we define the **standard free resolution of** $B$, $P_A(B) \to B$ (or $P(B)$ if $A$ is clear from the context), by the standard simplicial resolution defined by $(A[-], \mathrm{forget})$. We remark that this resolution is functorial in the nicest possible ways, and we will use many of these properties without restating them or proving them, instead saying "by functoriality of the standard resolution ...," see §II.1.2 of [1].

**Remark.** This resolution is explicitly given by $P_A(B)^1 = A[B]$, $P_A(B)^2 = A[A[B]], \ldots$.

Now, we need to discuss technicalities of the derived category $D.(A)$. Unfortunately, there isn't a way to prove the fundamental theorems of deformation theory without addressing these. Let $\Delta(n)$ denote the simplicial set represented by $[n] \in \Delta$. Define

$$\begin{aligned} \gamma &= \mathbb{Z}_{\Delta(1)} / \operatorname{im} \mathbb{Z}_{d^0} \\ \sigma &= \mathbb{Z}_{\Delta(1)} / \operatorname{im} \mathbb{Z}_{d^0} + \operatorname{im} \mathbb{Z}_{d^1}, \end{aligned}$$

where $d^0$ and $d^1$ are the two injections $[0] \to [0,1]$. Note that the notation is as before: $\mathbb{Z}_{\Delta(1)}$ is the free $\mathbb{Z}$-module generated by $\Delta(1)$. We have an exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\mathbb{Z}_{d^1}} \gamma \to \sigma \to 0, \tag{11}$$

split in each degree.

Let now $A$ be a simplicial ring. For $E \in A\text{-mod}$, set $\gamma E = \gamma \otimes_{\mathbb{Z}} E$ and $\sigma E = \sigma \otimes_{\mathbb{Z}} E$, where $\gamma$ and $\sigma$ are regarded as constant simplicial $\mathbb{Z}$-modules. Tensoring Equation 11 with $E$ gives an exact sequence

$$0 \to E \xrightarrow{i_E} \gamma E \to \sigma E \to 0.$$

These functors are generalizations of the functors of "cone" and "suspension," respectively, to the derived category, and we will call them by these names. $\gamma$ and $\sigma$ admit right adjoints which we denote by $\varphi$ and $\omega$, respectively. It is a fact that $\sigma$ and $\omega$ preserve homotopies and quasi-isomorphisms (the statement for $\sigma$ follows from the SES above). It follows that they extend to functors also denoted by $\sigma, \omega$ from $D.(A) \to D.(A)$, and it is a fact that $\sigma : D.(A) \to D.(A)$ is fully faithful [Corollary I.3.2.1.10 Illusie]. We then define

$$\mathbb{E}xt^p_A(E, F) = \operatorname*{colim}_{n \geq -p} \operatorname{Hom}_{D.(A)}(\sigma^n E, \sigma^{n+p} F),$$

where $E, F \in D.(A)$. Note that it follows immediately from the definition that

$$\begin{aligned} \mathbb{E}xt^p(\sigma^i E, F) &\cong \mathbb{E}xt^{p-i}(E, F) \\ \mathbb{E}xt^p(E, \sigma^i F) &\cong \mathbb{E}xt^{p+i}(E, F). \end{aligned}$$

**Example.** Suppose $A$ is a trivial simplicial ring with essentially constant value $\Lambda$ and that we identify $D.(A)$ with $D(\Lambda)$. In this case, $\sigma$ just becomes the shift functor, $X \mapsto X[1]$, and the above $\mathbb{E}xt$ identifies with the usual hyperext.

For $v$ a morphism in $C(A\text{-mod})$, denote by $C(v)$ the usual cone over $v$. Let $X = (0 \to E \xrightarrow{u} F \xrightarrow{v} G \to 0)$ be an exact sequence of $A$-modules. Associate to $X$ an exact triangle $\boldsymbol{\chi}(X)$ of $D.(A)$ as follows. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \xrightarrow{-i_E} & \gamma E & \longrightarrow & \sigma E & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \mathrm{id}} & \begin{bmatrix}-i_E\\ u\end{bmatrix} & \uparrow{\scriptstyle \mathrm{pr}_1} & & \uparrow{\scriptstyle p} & & \\ 0 & \longrightarrow & E & \longrightarrow & \gamma E \oplus F & \longrightarrow & C(u) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle s} & & \\ 0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 0 \end{array},$$

where the map $s$ is a quasi-isomorphism. Set

$$\boldsymbol{\chi}(X) = (E \xrightarrow{u} F \xrightarrow{v} G \xrightarrow{ps^{-1}} \sigma E).$$

This is clearly a distinguished triangle depending functorially on $X$, and, moreover, any distinguished triangle in $D.(A)$ is isomorphic to one of the form $\boldsymbol{\chi}(X)$. Denote by $\chi(X)$ the morphism $G \to \sigma E$. Given a map $f : E \to E'$ (resp. $g : G' \to G$) of $A$-modules, denote by $f * X$ the pushout (resp. by $X * g$ the pullback) extension: $0 \to E' \to E' \overset{E}{\oplus} F \to G \to 0$ (resp. $0 \to E \to F \times_G G' \to G' \to 0$), where $E' \overset{E}{\oplus} F$ denotes the obvious pushout. By definition, we have

$$\chi(f * X) = (\sigma f)\chi(X)$$
$$\chi(X * g) = \chi(X)g.$$

Moreover, $\chi$ induces a functorial homomorphism

$$\chi : \mathrm{Ext}^1(G, E) \to \mathbb{E}xt^1(G, E),$$

where the $\mathrm{Ext}$ is calculated in the abelian category $A$-mod. We can thus use $\chi$ to obtain a description of elements of the RHS in terms of extensions of $A$-modules.

The final piece of technology is derived tensor products. We want to be able to write down a functor $- \overset{\ell}{\otimes} - : D.(A) \times D.(A) \to D.(A)$, which when restricted to $\mathrm{Hot.}(A)$ (the category of $A$-modules up to homotopy), is the left derived functor of the tensor product in $\mathrm{Hot.}(A)$. For the details regarding homotopies in (simplicial) categories, please see §§1.1.5-6 of [1] (or elsewhere). We will approach this problem by defining it as the composition of two functors, and then proving that the functor we have derived is the left derived functor. To begin, note that we have functors $A\text{-mod} \to A\text{-mods} \xrightarrow{\Delta} A\text{-mod}$, where $A\text{-mods} := \mathrm{Simpl}(A\text{-mod})$. Moreover, their composition is the identity. Indeed, the first functor associates to an $A$-module $L$ the vertically trivial simplicial $A$-module with value $L$, while the second associates to a simplicial $A$-module $M$ the diagonal subobject $\Delta M$. Denote by $D._v(A)$ (resp. $D._{hv}(A)$) the category $A$-mods localized with respect to morphisms which induce isomorphisms (resp. quasi-isomorphisms) on the homology $A$-modules. The spectral sequence of bicomplexes $H_p^h H_q^v \Rightarrow H_*$ along with Eilenberg-Zilber gives that $\Delta$ factorizes: We have the commutative diagram

$$\begin{array}{ccccc} A\text{-mods} & \longrightarrow & D._v(A) & \longrightarrow & D._{hv}(A) \\ \downarrow{\scriptstyle \Delta} & & & & \downarrow{\scriptstyle \Delta} \\ A\text{-mod} & & \longrightarrow & & D.(A) \end{array},$$

where the horizontal arrows are localization functors. In particular, $\Delta$ induces for $E, F \in A\text{-mod}$ a functorial homomorphism

$$\chi : \mathrm{Ext}^n(E,F) \to \mathbb{E}xt^n(E,F)$$

for any $n \in \mathbb{Z}$. For $n = 1$, we recover the homomorphism for $\chi$ given above.

Define $D_v(A)$ (resp. $D_{hv}(A)$) as the localization of $C(A\text{-mod})$ by arrows which induce isomorphisms (resp. quasi-isomorphisms). By Dold-Puppe, $D_{\cdot_v}(A)$ identifies with the full subcategory of $D_v(A)$ formed by complexes that are acyclic in positive degrees (hence justifying the notation). Similarly, $D_{\cdot_{hv}}(A)$ identifies with the full subcategory of $D_{hv}(A)$ formed by complexes that are acyclic in posititve degrees. In the category $D_v^-(A)$ (i.e. complexes have bounded cohomology in the positive direction), the tensor product admits a left derived functor. We then have

**Lemma A1.** Let $L$ be an $A$-module, $f : E \to F$ a quasi-isomorphism of $A$-modules. If $L$ is flat, or if $E$ and $F$ are flat, then $L \otimes f : L \otimes E \to L \otimes F$ is a quasi-isomorphism.

*Proof.* If $A$ is trivial, the assertion results from Eilenberg-Zilber and Künneth. In the general case, tensor $f$ with the standard free resolution $F_A(L) \to L$ to obtain a commutative square of simplicial modules

$$\begin{array}{ccc} F_A(L) \otimes E & \longrightarrow & F_A(L) \otimes F \\ \downarrow & & \downarrow \\ L \otimes E & \longrightarrow & L \otimes F \end{array} \quad ,$$

where $L \otimes E$ and $L \otimes F$ are regarded as trivial simplicial $A$-modules. Now, the top horizontal arrow induces quasi-isomorphisms on each line, using the trivial case and the fact that if $X \in \mathrm{Simpl}(T)$, $M \in A\text{-mod}$, we have $A_X \otimes_A M \simeq \mathbb{Z}_X \otimes_Z M$, where $\simeq$ denotes "quasi-isomorphic." If $L$ is flat, or if $E$ and $F$ are flat, then the vertical arrows induce quasi-isomorphisms on each column. Then, we can use the spectral sequence of bicomplexes to find that the lower horizontal arrow also is a quasi-isomorphism. $\square$

Using Lemma A1, if $f, g$ are arrows in $D_v^-(A)$ that are invertible in $D_{hv}^-(A)$, then $f \overset{L}{\otimes} g$ is also invertible in $D_{hv}^-(A)$. In other words, the functor $\overset{L}{\otimes} : D_v^-(A) \times D_v^-(A) \to D_v^-(A)$ induces a functor

$$\overset{L}{\otimes} : D_{hv}^-(A) \times D_{hv}^-(A) \to D_{hv}^-(A),$$

which by restriction induces a functor

$$\overset{L}{\otimes} : D_{\cdot_{hv}}^-(A) \times D_{\cdot_{hv}}^-(A) \to D_{\cdot_{hv}}^-(A).$$

We now define

$$\overset{\ell}{\otimes} : D.(A) \times D.(A) \overset{\overset{L}{\otimes}}{\to} D_{\cdot_{hv}}(A) \overset{\Delta}{\to} D.(A). \tag{12}$$

**Proposition A2.** Let $E$ and $F$ be $A$-modules. The projective object of $D.(A)$ given by $\lim L \otimes M$ taken in the filtered category of quasi-isomorphisms $L \to E$, $M \to F$ of $\mathrm{Hot}.(A)$ is essentially constant with value $E \overset{\ell}{\otimes} F$. It is the left derived functor of $\otimes : \mathrm{Hot}.(A) \times \mathrm{Hot}.(A) \to \mathrm{Hot}.(A)$ in the sense of §1.4.4 of [1].

*Proof.* It suffices to observe that the quasi-isomorphisms $L \to E$ (resp. $M \to F$) with $L$ (resp. $M$) flat form a cofinal system (if $P$ is any $A$-module, $F_A(P) \to P$ induces, by Eilenberg-Zilber and a spectral sequence of bicomplexes, a quasi-isomorphism $\Delta F_A(P) \to P$, and $\Delta F_A(P)$ is a flat $A$-module); on the other hand, for $L$ (resp. $M$) flat we have $L \otimes F$ (resp. $E \otimes M$) $\simeq E \overset{\ell}{\otimes} F$. $\square$

**Remark.** If $A$ is trivial of value $\Lambda$, then $E \overset{\ell}{\otimes}_A F \simeq E \overset{L}{\otimes}_\Lambda F$, which follows from Eilenberg-Zilber.

Similarly, given a morphism $A \to B$ of simplicial rings of $T$, we can define

$$B \overset{\ell}{\otimes}_A - : D.(A) \xrightarrow{B \overset{L}{\otimes}_A -} D._{hv}(B) \xrightarrow{\Delta} D.(B),$$

and this functor is also a left derived functor of $B \otimes_A - : \mathrm{Hot.}(A) \to \mathrm{Hot.}(B)$; the proof is analogous to Proposition A2. We then have

**Proposition I.3.3.4.4.** Let $n \in \mathbb{Z}$. For $E \in D.(A)$, $F \in D.(B)$, there exists a canonical functorial isomorphism

$$\mathrm{Ext}^n_B(B \overset{\ell}{\otimes}_A E, F) \xrightarrow{\cong} \mathbb{E}xt^n_A(E, F). \tag{13}$$

*Proof Sketch.* It is enough to do the case $n = 0$. This is because we have a commutative diagram in categories of triangles and "true" triangles of $A$ and $B$, see §I.3.4.3 of [1]. We can suppose $E$ is flat, so we have $B \overset{\ell}{\otimes}_A E \to B \otimes_A E$. The canonical isomorphism

$$\mathrm{Hot}_B(B \otimes_A E, M) \cong \mathrm{Hot}_A(E, M)$$

gives, after passing to the limit over the quasi-isomorphismes $F \to M$, the desired isomorphism. $\square$

The final result is an extension of the above to simplicial rings. Let $A$ be a simplicial ring and $B$ an $A$-algebra. The standard free resolution $P_A(B)$ induces by passing to diagonal subobjects and by Eilenberg-Zilber and a spectral sequence of bicomplexes quasi-isomorphisms of $A$-algebras $\Delta P_A(B) \to B$. This algebra is free term-by-term, hence in particular flat as an $A$-module.

**Proposition (Künneth).** Let $B, C$ be $A$-algebras. The projective object of $D(A\text{-alg})$

$$B \overset{\ell}{\otimes}_A C := \lim P \otimes_A Q,$$

where the limit is taken over quasi-isomorphisms $P \to B$, $Q \to C$ of $\mathrm{Hot}(A\text{-alg})$, is essentially constant of value $P \otimes_A C$ (resp. $B \otimes_A Q$) for $P$ (resp. $Q$) flat over $A$. It follows that the functor $\overset{\ell}{\otimes}_A : D(A\text{-alg}) \times D(A\text{-alg}) \to D(A\text{-alg})$ is the left derived functor of $\otimes_A : \mathrm{Hot}(A\text{-alg}) \times \mathrm{Hot}(A\text{-alg}) \to \mathrm{Hot}(A\text{-alg})$.

For a proof, see I.3.3.5.2 of [1].