DW Theory

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1 Introduction

This note concerns various incarnations of Dijkgraaf-Witten (DW) theory in low dimensions, as well as a way to construct DW theory in any dimension $n \in \mathbb{N}$. DW theory was conceived as a finite gauge group generalization of Chern-Simons theory, which since then has served as a useful toy model for Chern-Simons and topological quantum field theories (TFTs) in general. Since the gauge group is finite, many spaces which are in usual Chern-Simons theory infinite-dimensional and unwieldy, such as the moduli space of flat connections on principal G-bundles on some "spacetime" 3-manifold M (here G is the gauge group). In DW theory, this moduli space becomes just $\mathrm{Map}(M, BG)$, which as we will see in §2 is "finite" in key ways, allowing it to be analyzed easily. As a taste, the idea is that the components of $\mathrm{Map}(M, BG)$ are themselves classifying spaces of finite groups.

We will focus on DW theory by using the cobordism hypothesis formulation of TFTs, attempting to give a rigorous definition and examples of calculations in DW theory in any dimension $n \in \mathbb{N}$. Before defining n-dimensional TFTs, we need to make a technical aside. Let M be a manifold of dimension $m \leq n$. An **n-framing of M** is a trivialization of the vector bundle $TM \oplus \underline{\mathbb{R}}^{n-m}$, where $\underline{\mathbb{R}}^{n-m}$ denotes the trivial bundle on M of rank $n-m$. Let $\mathsf{Bord}_n^{\mathrm{fr}}$ denote the bordism (∞, n)-category whose k-morphisms are given by n-framed k-manifolds, where $k \leq n$. An n-dimensional TFT is a symmetric monoidal functor

$$Z : \mathsf{Bord}_n^{\mathrm{fr}} \to \mathsf{C},$$

where C is some suitable symmetric monoidal (∞, n)-category. Note that the monoidal operation in the former category is just disjoint union. The cobordism hypothesis then states that the mapping $Z \mapsto Z(\mathrm{pt})$ gives a bijection between the set of isomorphism classes of symmetric monoidal functors $\mathsf{Bord}_n^{\mathrm{fr}} \to \mathsf{C}$ and the set of isomorphism classes of fully dualizable objects of C. In other words, we have

Theorem (Cobordism Hypothesis). The evaluation functor $Z \mapsto Z(\mathrm{pt})$ induces an equivalence

$$\mathsf{Fun}^{\otimes}(\mathsf{Bord}_n^{\mathrm{fr}}, \mathsf{C}) \to \widetilde{\mathsf{C}}$$

of $(\infty, 0)$-categories. Here, $\mathsf{Fun}^{\otimes}$ denotes the $(\infty, 0)$-category of symmetric monoidal functors between its arguments, which are (∞, n)-categories. $\widetilde{\mathsf{C}}$ denotes the underlying $(\infty, 0)$-category of C obtained by throwing out all non-invertible morphisms of C.

Remark. Although we will only be using it implicitly in this note, we will actually use a slightly modified version of the above theorem. Let G be a topological group acting continuously on a topological space X. Define the **homotopy fixed set** X^{hG} as the space of G-equivariant maps $\mathrm{Hom}_G(EG, X)$. Further, notice that the group $O(n)$ acts on $\mathsf{Bord}_n^{\mathrm{fr}}$ by change of framing; hence, it also acts on $\widetilde{\mathsf{C}}$.

Theorem 2.4.26 [7]. Cobordism Hypothesis for $\mathrm{SO}(n)$-manifolds. Let C be a symmetric monoidal (∞, n)-category with duals. There is then a canonical equivalence of (∞, n)-categories

$$\mathsf{Fun}^{\otimes}(\mathsf{Bord}_n^{\mathrm{SO}(n)}, \mathsf{C})) \to \widetilde{\mathsf{C}}^{h\,\mathrm{SO}(n)}.$$

In other words, we consider theories of *oriented* manifolds.

Although the rigorous definitions in this note will follow the framework of the cobordism hypothesis, we will throughout attempt to connect back to the physical picture. Originally, DW were attempting to construct finite gauge group theories which generalize the 3-dimensional Chern-Simons action

$$\frac{k}{8\pi^2} \int \mathrm{tr}(F \wedge F).$$

Such actions are classified in dimension $n \in \mathbb{N}$ by $\mathrm{H}^n(BG, \mathbb{R}/\mathbb{Z}) = \mathrm{H}^{n+1}(BG, \mathbb{Z})$ (equality is given by the homology sequence associated to $\mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z}$); see [1] for more details. Thus, DW reasoned that finite gauge group theories generalizing such actions will also be classified in this way. Just as Chern-Simons theory "counts G-bundles on 3-manifolds," DW theory was conceived as a physical theory where the partition function, i.e. our functor Z evaluated on n-manifolds, counts G-bundles, or rather G-bundle isomorphism classes. The reason we only want isomorphism classes is because we mod out by gauge equivalence in the path integral. Thus, whatever we assign to points, we better get that $Z(M^n)$ counts G-bundle isomorphism classes.

A G-bundle for a finite group G can be defined as follows. Take a covering map $P \to M$ and ask for a free G-action on P such that $P/G \cong M$. Further, the group G acts discretely since it is discrete, so it follows by Exercise I.7.7.5 from [3] that this data defines a map $\pi_1(M) \to G$. Recall that in order for this map to be well-defined, we choose base points in M and P. Further recall that if we don't choose a base point in P, then different choices will be the same map *up to conjugation by elements of* G (§I.7 [3]). Hence, isomorphism classes of G-bundles, i.e. equivalent maps $\pi_1(M) \to G$, are given by conjugacy classes of such maps under G-action. We should thus be able to reconcile our formal calculations of $Z(M)$ for n-manifolds M by just counting conjugacy classes of maps $\pi_1(M) \to G$. Physically, the partition function should assign vector spaces to closed $(n-1)$-manifolds N; these should be the Hilbert spaces of all possible states of theories that have a spacetime with boundary N. In particular, if M is an n-manifold with boundary N, then it will have associated to it a state vector which lives in $Z(N)$. Since our fields are just maps $\pi_1(M) \to G$, the quantum state vector should just be a balanced superposition of all possible states, i.e. over all possible field configurations. Each field has a weight, a probability, given by the reciprocal of the order of its automorphism group (i.e. of its gauge equivalence class). These considerations should also arise from the rigorous definitions which we will give.

2 1-dimensional DW Theory

In discussing DW theory, we will slowly work our way up via examples. Let's start with the most basic example, $n = 1$. In this case, our theory only has points and 1-manifolds. In particular, objects in our theory are just disjoint unions of points. Whether we choose to work with oriented or framed manifolds, in either case we end up with two versions of point: pt_+ and pt_-, corresponding to the two different choices of orientation or framing. The category C for DW theory in 1 dimension is just $\mathsf{Vect}_{\mathbb{C}}$, the category of complex vector spaces under tensor product, so to closed 1-manifolds we assign complex numbers and to 1-manifolds with boundary we assign vectors in $Z(\mathrm{pt}_+)$. Leaving this vector space unknown for now, notice that we can use the interval which connects pt_+ to pt_- as a map $\emptyset \to \mathrm{pt}_+ \sqcup \mathrm{pt}_-$ and vice-versa. Thus, we have that the interval gives a map

$$\mathbb{C} \to Z(\mathrm{pt}_+) \otimes Z(\mathrm{pt}_-)$$

or a map

$$Z(\mathrm{pt}_+) \otimes Z(\mathrm{pt}_-) \to \mathbb{C}.$$

There is thus a canonical bilinear pairing of $Z(\mathrm{pt}_-)$ with $Z(\mathrm{pt}_+)$; moreover, it is perfect. Indeed, this follows because we have the coevaluation $\mathbb{C} \to Z(\mathrm{pt}_-) \otimes Z(\mathrm{pt}_+)$, which gives us a composed map

$$\mathbb{C} \to Z(\mathrm{pt}_-) \otimes Z(\mathrm{pt}_+) \to \mathbb{C}.$$

Tensoring this with $Z(\mathrm{pt}_+)^\vee$ gives a map

$$Z(\mathrm{pt}_+)^\vee \to Z(\mathrm{pt}_+)^\vee \otimes Z(\mathrm{pt}_+) \otimes Z(\mathrm{pt}_-) \to Z(\mathrm{pt}_-),$$

and indeed this map must be inverse to the map $Z(\mathrm{pt}_-) \to Z(\mathrm{pt}_+)^\vee$ induced by the pairing. It follows that they are isomorphisms, so $Z(\mathrm{pt}_-) = Z(\mathrm{pt}_+)^\vee$. Now, $Z(S^1) = \dim_{\mathbb{C}} Z(\mathrm{pt})$, since we can view S^1 as a map

$$\emptyset \to \mathrm{pt} \sqcup \mathrm{pt} \to \emptyset,$$

so that $Z(S^1)$ is a map

$$\mathbb{C} \to V \otimes V \to \mathbb{C},$$

where $V = Z(\mathrm{pt})$. Identifying V with its dual, we can write $\mathrm{End}(V) = V \otimes V^\vee$, so that we are mapping to the identity endomorphism of V with the first map, and then taking its trace with the second, since V is fully dualizable, hence finite-dimensional. This number determines the theory completely up to isomorphism.

Now that we've gone through the generalities for $n = 1$ theories, we can begin DW theory proper. We will implement Z by writing it as a composition

$$Z : \mathsf{Bord}_1^{\mathrm{fr}} \to \mathsf{Fam}_1(\mathsf{Vect}_{\mathbb{C}}) \to \mathsf{Vect}_{\mathbb{C}},$$

where $\mathsf{Fam}_1(\mathsf{C})$ denotes the $(\infty, 1)$-category of finite groupoids X equipped with local systems $X \to \mathsf{C}$; similarly, morphisms are also equipped with functors to C. Recall that a local system is by definition a functor out of the fundamental groupoid of a space, where 1-morphisms are thought of as paths in the space, to some category. Homology with local coefficients (a generalization of homology with coefficients) can then be defined as a colimit of this functor (this just follows from the additivity properties of homology with respect to disjoint unions). Similarly, cohomology is a limit. Notice that in the case of a trivial local system, where morphisms are all sent to the identity and objects to a single object in the target category, taking a limit is just taking global sections with coefficients valued in the target object. We will exclusively focus on the case of trivial local systems in this note. A groupoid is **finite** if there is a finite number of inequivalent objects and each object has a finite automorphism group. In dimension 1, we can describe $\mathsf{Fam}_1(\mathsf{Vect}_{\mathbb{C}})$ explicitly: An object is a finite groupoid and a 1-morphism $C : X \to Y$ is a correspondence

$$\begin{array}{ccc} & C & \\ \swarrow & & \searrow \\ X & & Y \end{array}.$$

Composition of morphisms is by homotopy fiber product, and cartesian product gives us a symmetric monoidal structure. If we attach to each manifold M the finite groupoid of G-bundles on M, i.e. the space $X = \mathrm{Map}(M, BG)$ (it is finite since G is), we can upgrade this attachment to a functor

$$\mathrm{Bun}_G : \mathsf{Bord}_1^{\mathrm{fr}} \to \mathsf{Fam}_1,$$

where Fam_1 with no argument denotes the same category as before without the extra data of functors to C. Now, suppose we can lift Bun_G to a functor which fits into a commutative diagram

$$\begin{array}{ccc} \mathsf{Bord}_1^{\mathrm{fr}} & \xrightarrow{\quad I \quad} & \mathsf{Fam}_1(\mathsf{Vect}_{\mathbb{C}}) \\ & {\scriptstyle \mathrm{Bun}_G} \searrow \quad \swarrow & \\ & \mathsf{Fam}_1 & \end{array}$$

where the functor on the right is just the forgetful functor. Given such a functor I, it assigns not only the finite groupoid of G-bundles to a manifold but the groupoid along with a functor to $\mathsf{Vect}_{\mathbb{C}}$. Now, we can implement the partition function by "summing over G-bundles," i.e. via a functor

$$\mathrm{Sum}_1 : \mathsf{Fam}_1(\mathsf{Vect}_{\mathbb{C}}) \to \mathsf{Vect}_{\mathbb{C}}.$$

Formally, this functor will be a colimit: Given the groupoid $X = \mathrm{Map}(M, BG)$ and a functor $F : X \to \mathsf{Vect}_{\mathbb{C}}$, $\mathrm{Sum}_1(X) = \mathrm{colim}_{x \in X} F(x)$. Note that in $\mathsf{Vect}_{\mathbb{C}}$ we can evaluate this as either the limit or the colimit (they coincide). Our partition function is

$$Z = \mathrm{Sum}_1 \circ I.$$

Now, let $\lambda : G \to \mathrm{U}(1)$ be an abelian character. Note that λ comes from the physical description of DW theory. The generalization of the Chern-Simons action to 1d is given by formally replacing this action by a holonomy $\lambda(g)$, which we assign to principal G-bundles on 1-manifolds, i.e. on circles. Here, g is the image of 1 under the homomorphism $\pi_1(S^1) = \mathbb{Z} \to G$ which defines the bundle. It indeed classifies 1d DW theories, as it is a nontrivial theorem (see Theorem 3 below) that $\mathrm{Hom}(G, \mathrm{U}(1)) \cong \mathrm{H}^2(BG, \mathbb{Z})$. We demand that our theory sends the point

$$\mathrm{pt}_+ \mapsto (BG \to \mathsf{Vect}_{\mathbb{C}}),$$

i.e. the functor I takes the point to the functor from the groupoid $\mathrm{Map}(\mathrm{pt}_+, BG) = BG \to \mathsf{Vect}_{\mathbb{C}}$, thought of as an element of $\mathsf{Fam}_1(\mathsf{Vect}_{\mathbb{C}})$. This functor takes $* \mapsto \mathbb{C}$ and is the homomorphism λ on morphisms; explicitly, given an element $g \in G$, i.e. a morphism, $\lambda(g)$ acts on $\mathbb{C}$ in the natural way. We thus have an equivariant vector bundle over the point $G \to *$, which is given explicitly by the simplicial model for BG:

$$* \leftleftarrows G \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow}\; G \times G \cdots.$$

Taking the colimit will give the space of G-invariant sections of this bundle. Indeed, we are taking the colimit over the space $BG = */\!/G$ of the trivial local system (i.e. we set $\lambda \equiv 1$), so we get the vector space $\mathbb{C}$, which is exactly the 0'th equivariant (complex) homology group of BG. Note that it is also the cohomology by Poincaré duality; this is encoded in the fact that colimits and limits coincide in $\mathsf{Vect}_{\mathbb{C}}$. If $\lambda \not\equiv 1$, then G acts via λ and a section is invariant if and only if $\lambda \equiv 1$; otherwise, there are no invariant sections. It follows that the vector space is $\mathbb{C}$ if λ is trivial and 0 otherwise. This is all quite trivial, and follows from the fact that homology and cohomology satisfy strong additivity properties. For (co)homology with coefficients (in this case in $\mathbb{C}$), we have that

$$\begin{aligned} \mathrm{H}_n(X) &= \operatorname{colim} \mathrm{H}_n(X_\alpha) \\ \mathrm{H}^n(X) &= \lim \mathrm{H}^n(X_\alpha), \end{aligned}$$

where $X = \sqcup X_\alpha$ is a disjoint union of subspaces. In the case of BG, everything is trivial, and we are just assigning the vector space $\mathbb{C}$ everywhere. We will meet some more interesting examples below. Note that the point pt_- has the same vector space but with action by λ^{-1}. This is in accord with what the physical picture gives us, which we shall now describe. Physically, DW theory is given by a partition function which assigns to closed 1-manifolds a complex number, to the point a vector space, and to 1-manifolds with boundary an element of the vector space. The number it should assign to a closed 1-manifold is the number of the G-bundle isomorphism classes on it, weighted by their automorphisms, and the vector space to the point should be that generated by the isomorphism classes of G-bundles on it. Since all G-bundles on the point are isomorphic, we have that the vector space is just $\mathbb{C}$. Let's compute what happens on some 1-manifolds.

Example 1. Interval Between Two Points. First, consider an interval connecting two copies of a point. We have the following correspondence

$$\begin{array}{ccc} & \mathrm{Map}([0,1], BG) = BG & \\ \swarrow & & \searrow \\ \mathrm{Map}(0, BG) = BG & & BG = \mathrm{Map}(1, BG) \end{array},$$

where $[0,1]$ denotes the interval bordism and we no longer distinguish between point orientations since the vector spaces are isomorphic. Now, $[0,1]$ gives us a map $\mathbb{C} \to \mathbb{C}$, and this map is just multiplication by a number. We claim that this number is the number of G-bundles on $[0,1]$. This will follow from the functorial description of DW theory above. Starting with the G-bundle on the initial point, we pull it back to $\mathrm{Map}([0,1], BG)$ and then push it forward. But since $[0,1]$ is contractible, there is only one G-bundle isomorphism class on it. Thus, we are just multiplying the original vector space by 1 to go to the new vector space, i.e. the map $\mathbb{C} \to \mathbb{C}$ is just the identity. Alternatively, we have two local systems (one on each BG) which are the same; call them F as before. Since BG is connected, F has constant value equal to the vector space $\mathbb{C}$. For each $x \in \mathrm{Map}([0,1], BG)$, we have a map $\varphi(x) : F(p_1(x)) \to F(p_2(x)))$, where p_i are the projections—in this case isomorphisms. We thus have a map

$$\mathrm{Sum}_1(\mathrm{Map}([0,1], BG), \varphi) : \mathrm{Sum}_1(BG, F) \to \mathrm{Sum}_1(BG, F),$$

i.e. a map $\mathbb{C} \to \mathbb{C}$.

Proposition 1. This map is given by

$$\sum_{[x]\in\pi_0({p_1}^{-1}(*))} \frac{\varphi(x)}{|\operatorname{Aut}(x)|},$$

where the sum is taken over all equivalence classes of objects in the fiber of p_1 over $* \in BG$.

Proof. Notice that any fiber ${p_1}^{-1}(*)$ is mapped to $\mathbb{C}$ via $F \circ p_1$. Thus, to take the limit

$$\lim_{x\in X} \varphi(x),$$

where $X = \operatorname{Map}([0,1], BG)$, it suffices to take the limit of the maps φ over just one of these fibers; moreover, all fibers are identical. The limit is explicitly given by just composing the maps obtained by composing with φ the projections $\{p_b | b \in BG\}$ which are given by the limit $\operatorname{Sum}_1(BG, F)$, thus obtaining a map between the limits $\operatorname{Sum}_1(BG, F)$. All these projections are also the same map, the identity, since $Fg = \mathrm{id}$ for all $g \in G$. Now, since we have a local system, we may as well take the connected components of the fiber instead of its elements. So far, we've explained all pieces of the expression given except for the denominator $|\operatorname{Aut}(x)|$. The structure of the connected components of a given fiber are as follows. A G-bundle over the interval with value $*$ at the point 0 can have any value $g*$ at the point 1, and the number of non-isomorphic G-bundles, equivalently the number of non-homotopic maps $[0,1] \to BG$, $|\pi_0 \operatorname{Map}([0,1], BG)|$, will be given by the number of conjugacy classes in G. Now, the automorphism group of such a bundle will be its centralizer, as each bundle isomorphism class is given by its conjugacy class. Thus, we are reduced to taking a sum

$$\sum_{[g]\in\operatorname{conj}(G)} \frac{\mathrm{id}}{|C(g)|},$$

where $C(g)$ denotes the centralizer of g in G. But by Lagrange's theorem we have $|G/C(g)| = |[g]|$, so that this sum is the same as taking a sum over G and weighing it by $1/|G|$. Thus, we again get exactly one multiple of the identity map $\mathbb{C} \to \mathbb{C}$, hence the number 1. □

Remark. Notice that the proposition applies not just to this bordism, but to any general one. Moreover, it holds even more generally for a correspondence between arbitrary finite groupoids X and Y in $\mathsf{Fam}_1(\mathsf{Vect}_{\mathbb{C}})$.

This result also follows from the physical "counting G-bundles description." Indeed, we get a map $Z([0,1])$ from $Z(0) = \mathbb{C} \to \mathbb{C} = Z(1)$. To see its the identity, use the arguments at the beginning of the section to see that this map is the same map as the map $\mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$, which must be $1 \mapsto 1 \otimes 1$ (since we have the perfect pairing $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$).

Similarly, we have

Example 2. S^1**.** As before, $Z(S^1)$ gives us the trace of the identity endomorphism of $\mathbb{C}$, i.e. it gives us the number 1. But we can also understand this as the number of G-bundle isomorphism classes over the circle weighted by the reciprocal of the order of their automorphism group. Indeed, $\pi_1(S^1) = \mathbb{Z}$, and $\operatorname{Hom}(\mathbb{Z}, G) \cong G$ as sets; hence, there are $|G|$ G-bundles. Isomorphism classes of G-bundles are equivalently conjugacy classes of maps $\varphi \in \operatorname{Hom}(\mathbb{Z}, G)$, and $\operatorname{Aut}(\varphi) = C(\varphi)$, the centralizer of φ in G. So we take a sum over conjugacy classes of maps weighted by the orders of their automorphism groups, i.e. by the orders of their centralizers:

$$\begin{aligned} Z(S^1) &= \sum_{[g]\in \mathrm{conj}(G)} \frac{1}{|C([g])|} \\ &= \sum_{g\in G} \frac{1}{|G|} \\ &= 1 \end{aligned}$$

for $\lambda \equiv 1$. The second equality follows since the number of elements in a conjugacy class is equal to $|G|/|C([g])|$ (since the stabilizer of a class under conjugation is its centralizer). We'd like to reconcile this with the functorial description above. There is not much to do, since S^1 can be viewed as a map

$$\emptyset \to \emptyset,$$

so we have the following diagram.

$$\begin{array}{ccc} & \mathrm{Map}(S^1, BG) & \\ \swarrow & & \searrow \\ \mathrm{Map}(\emptyset, BG) = * & & * = \mathrm{Map}(\emptyset, BG) \end{array} .$$

As before, we assign the vector space $\mathbb{C}$ to the left point, pull it back to obtain a local system which is constant with value $\mathbb{C}$ on $X = \mathrm{Map}(S^1, BG)$, and then sum over all the components of $\pi_0(X)$ (dividing by their fundamental groups/automorphism groups) to obtain a number multiplying $\mathbb{C}$. But this number is just $|G|/|G| = 1$, so our map $\mathbb{C} \to \mathbb{C}$ is just multiplication by $1 = Z(S^1)$, the number of G-bundles on S^1 weighted by their automorphisms. Alternatively, this also follows from the equivariant vector bundle description. We are taking the limit of $\mathrm{H}^0([g];\mathbb{C})$, where $[g] \in \mathrm{conj}(G)$, over $\mathrm{conj}(G)$, to obtain the space of sections $\mathrm{H}^0(X;\mathbb{C})$. Equivalently, this can be rephrased in terms of the colimit of homology groups.

3 2-dimensional DW Theory

Now, let's move on to $n = 2$, where there is much more structure. The categories in question are $(\infty, 2)$-categories, which we will continue to just think of as ordinary 2-categories. Note that Fam_2 is just the 2-category of finite groupoids and correspondences where 2-morphisms are correspondences between 1-morphisms, i.e. $\mathsf{Fam}_k = \tau_{\leq k}\mathsf{Fam}$, where Fam is the ∞-category of finite groupoids and correspondences and $\tau_{\leq k}$ is the k-truncation functor. As before, we would like to implement the partition function Z as a composition of functors

$$Z : \mathsf{Bord}_2^{\mathrm{fr}} \xrightarrow{I} \mathsf{Fam}_2(\mathsf{Alg}) \xrightarrow{\mathrm{Sum}_2} \mathsf{Alg},$$

where Alg is the 2-category of algebras, bimodules, and intertwiners. Composition of bimodules is given by tensor product over the "middle" algebra. Composition of bimodules is given by tensor product over the "middle" algebra. As before, Sum_2 is just a colimit (or limit; they coincide) of the local system over the groupoid assigned to a bordism. We claim that

$$Z(\mathrm{pt}) = \mathbb{C}[G],$$

the complex group algebra of G. Note that this can indeed be viewed as a category: A complex algebra is an object in the 2-category Alg; thus, it together with its bimodules form a category. Let's see how we can obtain this from the formalism developed in the previous section. The groupoid in question is as before BG, and it comes equipped with a local system F valued in Alg, the 2-category described above. As before, to the point $*$ is assigned $\mathbb{C}$, now thought of as an algebra object in this category. To a 1-morphism $g \in G$, we assign the $(\mathbb{C}, \mathbb{C})$-bimodule $\mathbb{C}_g = \mathbb{C}$, and to the identity 2-morphisms in BG we assign the identity map. Note that this is the trivial local system; there are no interesting 1- or 2-morphisms. Indeed, this corresponds to the trivial element of $\mathrm{H}^3(BG, \mathbb{Z})$, which classifies central extensions of G by $\mathrm{U}(1)$. A nontrivial central extension will give a twisted group algebra, where there are now isomorphisms $\mathbb{C}_g \otimes \mathbb{C}_h \to \mathbb{C}_{gh}$ which in general depend on g and h. Similarly, associativity will no longer be on the nose, but only up to associator isomorphisms. We will not consider this in this note; see [2] for details. Then, $Z(\mathrm{pt}_+)$ is the 2-colimit over BG of this functor, which can again be computed as the 2-limit: They are both just (suitable generalizations of) direct sum in this category (since our groupoids are *finite*). Recall that a **2-limit** is a limit where we require that the cones commute only up to 2-isomorphisms and the universal property gives an *equivalence of categories* rather than an isomorphism of sets. Let's get the answer $Z(\mathrm{pt}) = \mathbb{C}[G]$ by calculating this limit.

Proposition 2. $\lim_{b \in BG} F(b) = \mathbb{C}[G]$.

Proof. We will first show that $\mathbb{C}[G]$ forms a cone over F. Given the object $* \in BG$, it is mapped to $\mathbb{C}$ via F, and we have a map $\mathbb{C}[G] \to \mathbb{C}$. Given $g \in G$ viewed as a map $* \mapsto *$, we need to show that the mapping $\mathbb{C}[G] \to \mathbb{C}$ associated to $g*$ is the same as that associated to $*$ composed with the mapping $Fg = \mathbb{C}_g$. But indeed, tensoring with the bimodule $\mathbb{C}$ over $\mathbb{C}$ gives the original $(\mathbb{C}, \mathbb{C}[G])$-bimodule we started with, so that we have one map $\mathbb{C}[G] \to \mathbb{C}$ in Alg. Thus, we do indeed have a cone over F. We now need to show that it is universal, which in the 2-categorical sense means that the category of $(\mathbb{C}[G], A)$-bimodules for any other cone $A \in \mathsf{Alg}$ is *equivalent* to a category of $(\mathbb{C}_g, A)$-bimodules. Suppose given another cone $A \in \mathsf{Alg}$ and a family of $(\mathbb{C}, A)$-bimodules parametrized by BG. Given one of these bimodules, we construct a $(\mathbb{C}[G], A)$-bimodule by asking that the original $\mathbb{C}$-action now acts according to the $(\mathbb{C}, \mathbb{C}[G])$-bimodule action which we assign to $\mathbb{C}[G]$. Said differently, for any other cone A, the category of $(\mathbb{C}[G], A)$-bimodules is equivalent to the category of (right) A-modules N along with a family of A-module isomorphisms $\mathbb{C}_g \otimes N \cong N$. It is this family which encodes the different actions of $\mathbb{C}$ on N induced by its left $\mathbb{C}[G]$-module structure given by the cone $\mathbb{C}[G]$, hence giving us a $(\mathbb{C}[G], A)$-bimodule exhibiting $\mathbb{C}[G]$ as the limit in question. □

The cobordism hypothesis asserts, and we will now show, that we can recover the value of Z on any higher-dimensional manifold. Let's start with 1-manifolds. For any 1-manifold M, $Z(M)$ is indeed given by a vector space, as we can view it as a morphism from the empty set to itself. As before, $Z(\emptyset) = \mathbb{C}$, since the empty collection of 1-manifolds is a unit object in $\mathsf{Bord}_2^{\mathrm{fr}}$ and $\mathbb{C}$ is the unit object under tensor product of algebras. Now, since $Z(\emptyset) = \mathbb{C}$, $Z(M)$ is then a $(\mathbb{C}, \mathbb{C})$-bimodule, in other words, a complex vector space. We would like an analog of Proposition 1 which will allow us to compute $Z(M)$ as a certain (1-categorical) limit. The 1-dimensional story goes through in a similar way here: Given the algebras associated to points, $\mathbb{C}[G] \in \mathsf{Alg}$, we can view the path integral on the interval $[0, 1]$ as a map $\mathbb{C}[G] \to \mathbb{C}[G]$, i.e. as a $(\mathbb{C}[G],\mathbb{C}[G])$-bimodule. Explicitly, for each $x \in X = \mathrm{Map}([0,1],BG) \in \mathsf{Fam}_2(\mathsf{Alg})$, we have a map $\varphi(x): F(p_1(x)) \to F(p_2(x))$. As before, the path integral $Z([0,1])$, i.e. the bimodule in question, is computed as a limit

$$\mathrm{Sum}_2(X,\varphi) = \lim_{x\in X} \varphi(x).$$

Notice that this limit is now taken in the category of $(\mathbb{C}[G],\mathbb{C}[G])$-bimodules. We take the limit over X, really over $\pi_0 X$, which as before has $|\mathrm{conj}(G)|$ elements. Thus, we get that we obtain a single map φ as before which is the identity, i.e. we obtain the $(\mathbb{C}[G],\mathbb{C}[G])$-bimodule $\mathbb{C}[G]$ (since tensor product *over* $\mathbb{C}[G]$ should give us back the original bimodule). The sum we obtain is exactly the same as in Example 1.

As before, we have

Example 3. S^1**.** We would like to compute $Z(S^1)$ as a 1-categorical limit, but, before doing so, let's see how to get the vector space by thinking about the partition function from the physical point of view. G-bundles on S^1 up to isomorphism are given by conjugacy classes of maps $\pi_1(S^1) \to G$. Now, $\mathrm{Hom}(\mathbb{Z},G) \cong G$ as sets, so $Z(S^1)$ is the vector space generated by conjugacy classes of G; equivalently, this is the space of class functions on G, or maps $G \to \mathbb{C}$ up to conjugation. The representation theory of finite groups tells us that this vector space is

$$\mathrm{Rep}(G) \otimes_{\mathbb{Z}} \mathbb{C},$$

where $\mathrm{Rep}(G)$ is the representation ring of G; see Proposition 2.30 and the comments below it in [4].

The limit definition gives us exactly the same answer, since we are asked to take a limit over the space $\pi_0 X = \pi_0 \mathrm{Map}(S^1, BG)$, but X has exactly one connected component for each G-bundle isomorphism class, so we get a sum over $(\mathbb{C},\mathbb{C})$-bimodules $\mathbb{C}$, and we count each $\mathbb{C}$ as many times as there are connected components in X. We thus get the same answer: the $(\mathbb{C},\mathbb{C})$-bimodule, or vector space,

$$\mathbb{C}[G]^G,$$

the conjugation invariant elements of $\mathbb{C}[G]$.

There is another way to think about $Z(S^1)$, namely as the composition $\emptyset \to \mathrm{pt}_+ \sqcup \mathrm{pt}_- \to \emptyset$. We thus have that $Z(S^1)$ is the tensor product of a $(\mathbb{C}[G]\otimes\mathbb{C}[G]^{\mathrm{op}},\mathbb{C})$-bimodule with a $(\mathbb{C},\mathbb{C}[G]\otimes\mathbb{C}[G]^{\mathrm{op}})$-bimodule. Note that these bimodules are $\mathbb{C}[G]$ again, as the argument above for $Z([0,1])$ shows. Composition of bimodules is given by tensor product over the middle factor, so we get that

$$Z(S^1) = \mathbb{C}[G] \underset{\mathbb{C}[G]\otimes\mathbb{C}[G]^{\mathrm{op}}}{\otimes} \mathbb{C}[G].$$

Recalling that the opposite algebra is given by multiplication in the reverse direction, we get that this is exactly the quotient of $\mathbb{C}[G]$ by the subspace generated by all commutators; hence, it is exactly $Z(S^1)$ as obtained above, $\mathbb{C}[G]^G$.

Let's now do a surface.

Example 4. $S^1\times S^1$**.** Consider the torus $T = S^1 \times S^1$. Physically, counting G-bundles on the torus up to isomorphism is counting conjugacy classes of maps $\pi_1(T) = \mathbb{Z}\oplus\mathbb{Z} \to G$. Explicitly, the number is

$$\sum_{\mathrm{conj}(G)\ni[\varphi]:\mathbb{Z}\oplus\mathbb{Z}\to G} \frac{1}{|\mathrm{Aut}(\varphi)|} = \sum_{\substack{([g],[g'])\in\mathrm{conj}(G)\times\mathrm{conj}(G)\\ [g,g']=1}} \frac{1}{|C(\{g,g'\})|} = \sum_{\substack{g,g'\in G\\ [g,g']=1}} \frac{1}{|G|},$$

where the second equality follows by Lagrange's theorem as before. We would like to obtain this number by starting from the group algebra $\mathbb{C}[G]$. Following our work in Example 2, we again get a correspondence in $\mathsf{Fam}_2(\mathsf{Alg})$, and this gives us in an exactly analogous way a map $\mathbb{C} \to \mathbb{C}$ which is the limit of the various maps $\varphi(x) : F(p_1(x)) \to F(p_2(x))$ associated to $x \in X = \mathrm{Map}(T, BG)$ and the local systems F (now valued in Alg) on BG. But indeed this limit just counts G-bundles up to isomorphism by definition of X, as before, weighting them with the order of their automorphism group G; the argument of Proposition 1 goes through in the same way. Note that indeed $Z(T) = \dim_{\mathbb{C}} Z(S^1)$, as

$$\sum_{\substack{g,g' \in G \\ [g,g']=1}} \frac{1}{|G|} = \sum_{g \in G} \frac{|C(g)|}{|G|} = \sum_{[g] \in \mathrm{conj}(G)} 1.$$

This is something expected from a TFT, namely that $Z(S^1 \times M) = \dim_{\mathbb{C}} Z(M)$.

It is now straightforward to compute other surfaces.

Example 5. Genus g Surface. Let Σ_g be a genus g connected surface. Then its fundamental group is just $\pi_1(\Sigma_g) = \oplus_{i=1}^{g+1} \mathbb{Z}$. Hence,

$$Z(\Sigma_g) = \sum_{\substack{g_1, \ldots, g_{g+1} \in G \\ [g_i, g_j] = 1 \ \forall i,j \in \{1, \ldots, g+1\}}} \frac{1}{|G|}.$$

As a special case, if $g = 0$, so $\Sigma_0 \sim S^2$, then $Z(S^2) = 1$. Physical considerations suggest that $Z(S^1 \times M)$ for M a 1-manifold should give the dimension of the vector space $Z(M)$. Since 1-manifolds are just disjoint unions of circles, this example should cover all such dimensions.

Lastly, let's do some surfaces with boundary, say

Example 6. $S^1 \times S^1$ with a Hole. Let M now be the torus with a hole. Our TFT should assign a vector in the vector space $Z(S^1) = Z(\partial M)$. Starting with the physical description, this vector should be given by maps $\pi_1(M) \to G$, where $\pi_1(M) = \langle a, b \rangle$ is just the free group on two generators (the relation $[a,b] = 1$ no longer needs to be satisfied, as the 2-cell from which it arises can no longer be traversed in the manner $aba^{-1}b^{-1}$). We thus get that isomorphism classes of principal G-bundles on M are the same as giving two elements of G up to conjugation. Hence,

$$Z(M) = \sum_{([g],[g']) \in \mathrm{conj}(G) \times \mathrm{conj}(G)} \frac{[gg']}{|C(\{g, g'\})|}$$

which is a vector in $\mathbb{C}[G]^G = Z(S^1)$. The reason we take the multiple $[gg']$ is because we really want to take a vector in $Z(S^1)$ that corresponds to the object $[g] \otimes [g']$. The multiplication map then gives us the result.

We should be able to get the same answer by thinking about M as a map $\emptyset \to S^1$, so that $Z(M)$ is a homomorphism of $(\mathbb{C}, \mathbb{C})$-bimodules, i.e. of vector spaces, $\mathbb{C} \to \mathbb{C}[G]^G$. It is thus a linear map, so its value is completely determined by where it sends $1 \in \mathbb{C}$, and we claim that this vector is the vector $Z(M)$ we found above. We should think of this map as a correspondence

$$\begin{array}{ccc} & \mathrm{Map}(M, BG) & \\ \swarrow & & \searrow \\ * & & \mathrm{Map}(S^1, BG) \end{array}.$$

In this case, we need to figure out what the maps $\varphi(x) : \mathbb{C} \to \mathbb{C}[G]^G$ are, where $x \in X$, associated to the correspondence above. Then, we should just take the limit $\lim_{x \in X} \varphi(x)$. For this, it is easiest to go one step further, and think instead of M as a map $\emptyset \to S^1 \sqcup S^1 \to S^1$. The first map we have already met in Example 4. Namely, the torus can be thought of as a map $\emptyset \to S^1 \sqcup S^1 \to \emptyset$, so that the result from that example should be obtained as the image of 1 under the composition

$$\mathbb{C} \to \mathbb{C}[G]^G \otimes \mathbb{C}[G]^G \to \mathbb{C}.$$

The last map is just the trace pairing on the underlying algebra $\mathbb{C}[G]$, so that

$$1 \mapsto \sum_{([g],[g']) \in \mathrm{conj}(G) \times \mathrm{conj}(G)} \frac{[g] \otimes [g']}{|C(\{g, g'\})|}$$

gives the first map. The condition that g, g' commute is given by the fact that the trace pairing vanishes on commutators, which gives another way to derive the result of Example 4. We can now compute what happens in the case of M, since the second map in its composition is just multiplication in $\mathbb{C}[G]^G$, inherited from the natural multiplication on $\mathbb{C}[G]$. Hence, we have

$$\begin{aligned} \mathbb{C} \to & \mathbb{C}[G]^G \otimes \mathbb{C}[G]^G \to \mathbb{C}[G]^G \\ 1 \mapsto & \sum \frac{[g] \otimes [g']}{|C(\{g, g'\})|} \mapsto \sum \frac{[gg']}{|C(\{g, g'\})|}. \end{aligned}$$

The individual maps $\varphi(x)$ are just the individual pieces of the sum, and taking the limit sums up all of the maps weighted by their automorphism groups, the denominators.

4 3-dimensional DW Theory and the General Case

Rather than begin with the 3-dimensional story as we have done in the previous two sections, we will instead describe how to construct DW theory (at least in principle) in any dimension $n \in \mathbb{N}$ and then show how the 3-dimensional DW theory is a special case of this construction, focusing on $Z(\mathrm{pt})$ and $Z(S^1)$. We need to define higher algebra objects which encapsulate the constructions of the previous two sections. We define an m-**algebra** as an algebra object in the symmetric monoidal m-category of $(m-1)$-algebras. Morphisms between such m-algebras are then bi-module objects in the category of $(m-1)$-algebras. A 0-**algebra** is a complex vector space, so that morphisms between 0-algebras are linear maps. The 2-category Alg is the 2-category of 1-algebras, familiar from the previous section. We claim that in any dimension $n \in \mathbb{N}$, the target of the classical theory, i.e. the codomain of I, $\mathsf{Fam}_n(\mathsf{C})$ has $\mathsf{C} = \mathsf{Alg}_{n-1}$, the n-category of $(n-1)$-algebras. Indeed, for $n \in \{1, 2\}$ this is the case.

Example 7. 2-algebras. A 2-algebra is an algebra A together with an $(A, A \otimes A)$-bimodule M which defines the 2-multiplication and a left A-module E which defines the identity. There are associativity and unit intertwiners, 2-morphisms in the category of algebras, which satisfy compatibility rules, see [2] for details. A morphism of 2-algebras $A \to B$ is then a (B, A)-bimodule in the category of algebras—an algebra N and an $(N, B \otimes N)$-bimodule P along with an $(N, N \otimes A)$-bimodule Q with compatibility morphisms which give the bialgebra structure. The key facet of this construction is that the 2-algebra structure on A defines a monoidal structure on A-mod, the category of left A-modules. In particular, the module functor $A \mapsto A\text{-mod}$ is fully faithful, so that any 2-algebra can be equivalently thought of as a monoidal category, and this is what people usually do when they think of 3-dimensional DW theory. For example, given a Hopf algebra A, it becomes a 2-algebra under the multiplication $A \otimes A \in A\text{-mod}$. We can write $A = \mathbb{C}(G)$ for the Hopf algebra of a finite group with pointwise multiplication and comultiplication induced from the Hopf structure, then the category A-mod is the category of vector bundles over G. Thus, the monoidal category defined by the Hopf 2-algebra is precisely the category $\mathsf{Vect}[G]$: Its objects are complex vector bundles over G and morphisms are linear vector bundle maps. It is naturally a fully dualizable object in a certain 3-category, and this 3-category is exactly A-mod. The monoidal structure is defined by convolution of vector bundles in the following way.

To define a 3-dimensional DW theory, we need to generalize the Chern-Simons action, and such generalizations are classified by elements in $\mathrm{H}^4(BG; \mathbb{Z}) = \mathrm{H}^3(BG; \mathbb{R}/\mathbb{Z})$. Define a **2-cocycle on G with values in hermitian lines** as a pair (K, θ) consisting of a hermitian line bundle $K \to G \times G$, for each triple $x, y, z \in G$ an isometry

$$\theta_{x,y,z} : K_{y,z} \otimes K_{xy,z}^{-1} \otimes K_{x,yz} \otimes K_{x,y}^{-1} \to \mathbb{C},$$

and a cocycle condition

$$\theta_{y,z,w}\theta_{xy,z,w}^{-1}\theta_{x,yz,w}\theta_{x,y,zw}^{-1}\theta_{x,y,z} = 1$$

for each quadruple $x, y, z, w \in G$.

Remark. Recall that group cohomology with coefficients in an abelian group A can be defined as

$$\mathrm{H}^n(G; A) = \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A),$$

where we view $\mathbb{Z}$ and A as $\mathbb{Z}[G]$-modules. However, there is another definition, where group cohomology can be defined as the set of homomorphisms from $G^n \to B^{n-1}A$, obtained from the simplicial model for $BG = *//G$ (see `nLab`). From here it becomes clear that a 2-cocycle is a line bundle over $G \times G$ with these conditions. Notice that this immediately gives us the following theorem, which explains why 1-dimensional DW theories are classified by abelian characters. The same result can be obtained by noting that the chain complex associated to $A[BG]$, where BG denotes the simplicial set $*//G$, is the canonical chain complex used to compute the group (co)homology with coefficients in A. In particular, we have formulas

$$\mathrm{H}_*(G; A) \cong \mathrm{H}_*(BG; A) = \mathrm{H}_*(|BG|; A),$$
$$\mathrm{H}^*(G; A) \cong \mathrm{H}^*(BG; A) = \mathrm{H}^*(|BG|; A),$$

where $|BG|$ is the geometric realization of the simplicial set BG. See §8.2 of [10] for more details.

Theorem 3. $\mathrm{Hom}(G, \mathrm{U}(1)) = \mathrm{H}^2(BG; \mathbb{Z})$.

Proof. Since $\mathrm{H}^2(BG;\mathbb{Z}) = \mathrm{H}^2(G;\mathbb{Z})$ by definition, and $B\mathbb{Z} = \mathrm{U}(1)$, we have that $\mathrm{H}^2(BG;\mathbb{Z})$ is given by $\mathrm{Hom}(G,\mathrm{U}(1))$. □

Proposition 4. $\mathrm{H}^3(BG;\mathbb{R}/\mathbb{Z})$ is the set of isomorphism classes of 2-cocycles (K,θ) on G with values in hermitian lines.

Proof. Choosing $k_{x,y} \in K_{x,y}$ of unit norm, the object

$$\omega_{x,y,z} = \theta_{x,y,z}(k_{y,z}k^{-1}_{xy,z}k_{x,yz}k^{-1}_{x,y})$$

is a 3-cocycle with values in $\mathbb{R}/\mathbb{Z}$. The resulting element of $\mathrm{H}^3(BG;\mathbb{R}/\mathbb{Z})$ is independent of the choices $\{k_{x,y}\}$ and of the representative (K,θ) in an equivalence class. Indeed, the isometries θ are equivalently isometries of $\mathbb{R}^2$, so they are rotations, reflections, and/or translations. But these cannot depend on the choices of *unit norm* elements in the argument. Similarly, another representative (K',θ') in the equivalence class of (K,θ) has different isometries in general, but if it is isomorphic to (K,θ), then the isometries themselves differ by an isometry of $\mathbb{C}$. This will change the cocycle by this isometry, i.e. if $\theta' = \eta\theta$, then $\omega' = \eta\omega$. But the cocycle condition gives that this isometry must be the identity; hence, $\omega = \omega'$. We now claim that this gives an isomorphism. Indeed, it is surjective, since given any 3-cocycle ω, we can take the 2-cocycle (K,θ) defined by $K_{x,y} = \mathbb{C}$ and $\theta_{x,y,z} = \omega_{x,y,z}$. It is furthermore injective, since inequivalent 2-cocycles will differ by a nontrivial isometry, hence will give different 3-cocycles. □

Finally, we can define the convolution in $\mathsf{Vect}[G]$. Fixing a 2-cocycle (K,θ) on G with values in hermitian lines, if $V,V' \to G$ are vector bundles, define

$$(V * V')_y = \bigoplus_{xx'=y} K_{x,x'} \otimes V_x \otimes V'_{x'}.$$

Thus, $\mathsf{Vect}[G]$ is a linear monoidal category, and it is this category which is assigned to a point in 3-dimensional DW theory. Explicitly, it is an object in the following 3-category C. Objects of C are $\mathbb{C}$-linear monoidal categories. Given a pair of such categories, A,A', a 1-morphism is an (A,A')-bimodule category, i.e. a $\mathbb{C}$-linear category B with a left action $\mathsf{A}\times\mathsf{B} \to \mathsf{B}$ and a right action $\mathsf{B}\times\mathsf{A}' \to \mathsf{B}$ which commute with each other up to coherent isomorphism. A 2-morphism $\mathsf{B} \to \mathsf{B}'$ is then a functor between bimodule categories—it must commute with the actions of A and A' up to coherent isomorphism. Finally, a 3-morphism between functors $F,F' : \mathsf{B} \to \mathsf{B}'$ is a natural transformation compatible with the isomorphisms inherent to F,F'.

Example 8. 3-algebras. A 3-algebra structure on A is the following. The 2-algebra structure on A makes it a monoidal category with a 2-category of modules A-mod. The 3-algebra structure gives a monoidal structure on the 2-category A-mod. In the case of Example 7 above, $\mathsf{Vect}[G]$-modules can be tensored over $\mathsf{Vect}[G]$ using the Hopf structure (note that this is only possible if we have not twisted, i.e. if the 2-cocycle is trivial).

We would now like to define the quantization map Sum_n for the classical theories defined above. Define **higher groupoids** as spaces, using the simplicial set model structure [7]. In this case, 0-groupoids are discrete sets, 1-groupoids are ordinary groupoids, i.e. $K(G,1)$'s. The idea is that we will linearize our spaces and turn them into higher algebras, i.e.

$$\mathrm{Sum}_n : \mathsf{Fam}_n(\mathsf{Alg}_{n-1}) \to \mathsf{Alg}_{n-1}.$$

The general definition is the following. The $(n-1)$-algebra associated to a connected space X has as its category of modules the $(n-1)$-category of local systems of $(m-1)$-algebras over X. In the case of a *based* space, this is the $(n-1)$-category of representations of ΩX on $(n-2)$-algebras; for disconnected spaces, we need only sum over components. In the sequel, let $m = n-1$ for notational convenience. Write $R_m(X)$ for this "groupoid algebra" of local systems.

Example 9. $n = 1$. $R_0(X)$ is the vector space of functions on $\pi_0 X$, which is exactly the space of invariant sections of the equivariant vector bundle over the point $G \to *$ obtained in §2.

Example 10. $n = 2$. We get the usual groupoid algebra, which becomes the group algebra $\mathbb{C}[G]$ after a choice of base points: It is the direct sum of the group algebras of the fundamental groups of the components of X. Note that this also follows from the discussion in §3. Notice that this is also a "vector space of sections" in the following sense. m-algebras are $(m+1)$-vector spaces, which the examples $m = 0, 1$ show. The groupoid X is incarnated as a $K(G,1)$, BG, thought of as an ∞-groupoid. Thus, in $n = 1$ we found that the equivariant vector bundle defined by BG, $G \to *$, can itself be viewed as a vector bundle over the category $\mathsf{Vect}_{\mathbb{C}}$. This fancy language just says that the local system with which BG is endowed allows us to quantize, i.e. give the sections of this bundle. In $n = 1$, this corresponds to just taking the sections of the original bundle $G \to *$. In $n = 2$, the same local system gives the sections of the bundle but now valued in the category $\mathsf{Vect}^2 = \mathsf{Alg}_1$, so the result is now a category built out of vector spaces.

Example 11. $n = 3$. Note that in the generalization above, our space X can be a homotopy 2-type. In particular, we can take X to be a $K(G,2)$ as a first example. It will quantize to the 2-algebra $\mathbb{C}[G]$ which is associated to the commutative group algebra $\mathbb{C}[G]$. The 2-algebra arises via its own multiplication map, and a morphism between such objects $A \to B$ is simply a $B \otimes A$-algebra. Note that they embed into monoidal categories by sending $\mathbb{C}[G]$ to the monoidal category with one object $*$ and $\mathrm{End}(*) = \mathbb{C}[G]$. In the case that X also has a π_1, i.e. a twisting, then we get a "crossed product 2-algebra;" call it A. We can use the monoidal structure on the category of modules of the underlying 1-algebra to describe A. As a 1-algebra, A consists of the functions on π_1 with values in the algebra $\mathbb{C}[G]$ with pointwise multiplication. Its linear category of modules is $\mathsf{Rep}(G)(\pi_1)$ which denotes bundles of G-representations over π_1. The monoidal category of A-modules is equivalent to $\mathsf{Rep}(G)$ as a linear category but carries a non-standard monoidal structure corresponding to convolution of characters. Now, use the fact that this category is dual (in the sense of Fourier or Pontryagin duality; see §12.3 of [8]) to the category $\mathsf{Vect}(G^*)$ of vector bundles on the Pontryagin dual group with the pointwise monoidal structure. The fact that it is pointwise follows from the Fourier dual description. π_1 then acts by automorphisms of the monoidal category $\mathsf{Vect}(G^*)$; if π_1 acts trivially, then $\mathbb{C}[G](\pi_1)$ is a Hopf algebra over $\mathbb{C}[G]$, giving a 2-algebra structure. Note that as in the previous example, we can again view the quantization procedure as giving sections of the 3-vector bundle over Vect^3 defined by the groupoid X.

We can now outline the general, inductive procedure for constructing $R_m(X)$. The vector space, i.e. 0-algebra, gives the finitely supported functions on the m-truncated homotopy

$$\coprod_{[x]\in\pi_0 X} \pi_m(X,x) \times \cdots \times \pi_1(X,x).$$

The 1-algebra is the $\mathbb{C}[\pi_m]$-valued functions on the union of the m-truncated homotopy groups with pointwise multiplication. The full m-algebra structure can be described via

$$R_m(X) = \bigoplus_{[x]\in\pi_0 X} R_{m-1}(\Omega_x X).$$

Notice that in the case of $n = 0+1$, $n = 1+1$, and $n = 2+1$, this subsumes the previous discussion.

Remark. Although we haven't considered it in this note, we can twist by cohomology classes in $\mathrm{H}^n(X;\mathbb{R}/\mathbb{Z})$. We get a projective cocycle for actions of ΩX on $(m-1)$-algebras, and the twisted group ring is the m-algebra with the same representation category. For example, in $m = 0$ we get a nontrivial action of π_1 on $\mathbb{C}$ for each component of X, and only the invariant lines are summed up to produce the vector space, cf. §2. If $m = 1$, the class gives a central extension of each π_1, which is quantized to the sum of the corresponding twisted algebras. We won't go further than this in describing twisting, though it is possible.

Let FH be the category of spaces with finitely many, finite homotopy groups with disjoint unions, products, and homotopy fiber products. Define the n-category $\mathsf{FH}_n = \tau_{\leq n}\mathsf{FH}$ as in §3. The assignment $X \mapsto R_{n-1}(X)$ can be upgraded into a symmetric monoidal functor

$$\mathrm{Sum}_n : \mathsf{FH}_n \to \mathsf{Alg}_{n-1},$$

generalizing the Sum_n defined at the beginning of this section. Although it is clear what it does on objects, it is not immediately clear what its action on morphisms is, because R_m is the Koszul dual functor of chains on the based loop space, so we use Koszul duality in the definition of Sum_n. This Koszul duality is exhibited by the following. The algebra of chains on the based loop space $\Omega_x X$ can be viewed as a differential graded coalgebra, assumed to be associative, via the cobar construction. On the other hand, the bar construction exhibits the algebra of chains on X as a differential graded algebra. These two are Koszul dual to one another [9]. Our map R_m is this duality functor. To a correspondence at level k, i.e. to a k-storied diagram of spaces, we assign the colimit of the diagram of group rings.

We can describe explicitly for 1-morphisms how this works. Suppose given a correspondence

$$\begin{array}{ccc} & C & \\ \swarrow & & \searrow \\ X & & Y \end{array},$$

where we assume that all our spaces are connected. If they're not, this is no issue, since different components are handled separately. Then to C we assign

$$\begin{aligned} \mathrm{Sum}_n(C) &= R_{n-1}(X) \underset{R_{n-1}(C)}{\otimes} R_{n-1}(Y) \\ &\cong R_{n-2}(F), \end{aligned}$$

where F is the homotopy fiber of $C \to X \times Y$. What this is saying is that we are free to compute Sum_n by computing the functor R in the theory *one level down*. The idea is that correspondences in an n-TFT can be viewed as correspondences between correspondences in $(n-1)$-TFTs. Note that what we obtain in this case is an $(n-2)$-algebra; the top multiplication layer has been used for the monoidal structure. The next example shows how this works.

Example 12. $Z(S^1)$ **in 3 Dimensions.** In Example 7, we claimed that $Z(\mathrm{pt}) = \mathsf{Vect}[G]$. We can compute $Z(S^1)$ for such a theory following the schematic of Example 3. As there, we can decompose S^1 as a map $\emptyset \to \mathrm{pt}_+ \sqcup \mathrm{pt}_- \to \emptyset$, where we remark that the opposite 2-algebra to $A = \mathsf{Vect}[G]$ is just the same 2-algebra with the reversed monoidal operation. We thus find, as in that example, that

$$Z(S^1) = A \underset{A \otimes A^{\mathrm{op}}}{\otimes} A.$$

We can also see this from the above formalism. Given a component of $X = \mathrm{Map}(S^1, BG)$, i.e. a conjugacy class in G, the homotopy fiber of the correspondence

$$\begin{array}{ccc} & \mathrm{Map}(S^1, BG) & \\ \swarrow & & \searrow \\ * & & * \end{array}$$

is just given by that component again (in general, the homotopy fiber of any connected space over the point space is just that connected space). Applying the functor one level down to this object, i.e. in the 2-dimensional theory, we get exactly the result above for $Z(S^1)$. Indeed, we get a sum

$$\bigoplus_{[g]\in\mathrm{conj}(G)} R_{n-2}(F_{[g]}) = \bigoplus_{[g]\in\mathrm{conj}(G)} \mathbb{C} = \mathbb{C}[G]^G.$$

Since on the level of 1-algebras $\mathbb{C}[G]^G = \mathbb{C}[G] \underset{\mathbb{C}[G]\otimes\mathbb{C}[G]^{\mathrm{op}}}{\otimes} \mathbb{C}[G]$, and $\mathbb{C}[G] = \mathbb{C}(G)$ as 1-algebras, we obtain the answer above yet again.

As claimed in Example 7, A is a fully dualizable object in the 3-category C (cf. there). Thus, there is an isomorphism $A \cong A^\vee = \mathrm{Hom}(A, \mathsf{Vect})$. Explicitly, this isomorphism is given by mapping a vector bundle over G to its fiber at the identity element, which corresponds to a bilinear form

$$\begin{aligned} A \otimes A &\to \mathsf{Vect} \\ V \otimes V' &\mapsto (V * V')_0. \end{aligned}$$

It thus follows that

$$\begin{aligned} Z(S^1) &\cong A \underset{A\otimes A^{\mathrm{op}}}{\otimes} A^\vee \\ &\cong \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, A). \end{aligned}$$

In particular, we see that Poincaré duality in this setting gives us an identification of the Hochschild cohomology of A with the Hochschild homology of A. Note that $Z(S^1)$ is also the **Drinfeld center of** A. This is the category whose objects are pairs (X, ε_X) consisting of $X \in A$ and a natural isomorphism $\varepsilon_X(-) : X \otimes - \xrightarrow{\cong} - \otimes X$ compatible with the monoidal structure:

$$\varepsilon_X(Y \otimes Z) = (\mathrm{id}_Y \otimes \varepsilon_X(Z)) \circ (\varepsilon_X(Y) \otimes \mathrm{id}_Z).$$

Lastly, note that this center is a braided tensor category. Namely, we have

Proposition 5. The center of A consists of twisted equivariant vector bundles $V \to G$, i.e. vector bundles with a twisted lifting of the G-action on G by conjugation:

$$L_{x,y} \otimes V_x \xrightarrow{\cong} V_{yxy^{-1}}.$$

Here, $L \to G \times G$ is the line bundle given by

$$L_{x,y} = K^{-1}_{yxy^{-1},y} \otimes K_{y,x}.$$

The cocycle condition on (K,θ) gives an isomorphism

$$L_{yxy^{-1},y'} \otimes L_{x,y} \xrightarrow{\cong} L_{x,y'y},$$

so L can be thought of as a line bundle over $\mathrm{Map}(S^1, BG)$, the groupoid formed by the G-action on itself by conjugation.

Proof. Suppose $V \to G$ is in the center. Let $V' \to G$ be the vector bundle which is the trivial line $\mathbb{C}_y$ at some $y \in G$ and zero elsewhere. Then the braiding gives an isomorphism

$$K_{y,x} \otimes \mathbb{C}_y \otimes V_x \xrightarrow{\cong} K_{yxy^{-1},y} \otimes V_{yxy^{-1}} \otimes \mathbb{C}_y.$$

By the definition of $L_{x,y}$, this gives the twisted lifting in the statement of the Proposition. □

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