Toën's Survey Sections 4.2-3

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0 Summary

Recall that given a smooth surjective morphism in Sch, $f : X \to Y$, we can recover Y as a derived stack by

$$Y \cong |X^\bullet/Y|.$$

We also have étale, faithfully flat, etc. descent. But DAG gives us another descent property, called "formal descent" for closed immersions of schemes. Let $f : X \to Y$ be a closed immersion of locally noetherian *classical* schemes. Define the formal completion of Y along X, $\widehat{Y}_X$, which is the stack represented by the formal scheme which is the formal completion of Y along X. We can explicitly describe this stack by

$$\widehat{Y}_X(R) := Y(R) \times_{Y(R_{\text{red}}} X(R_{\text{red}}),$$

where $R_{\text{red}} := \pi_0(R)_{\text{red}}$.

We then have the following characterization of $\widehat{Y}_X$.

Theorem 0.1 (Carlsson; Bhatt)**.** The augmentation morphism $X^\bullet/Y \to \widehat{Y}_X$ exhibits $\widehat{Y}_X$ as the colimit of the diagram $X^\bullet/Y$ inside the category of derived schemes. I.e. we have an equivalence

$$\mathsf{Map}_{\mathsf{Stk}}(\widehat{Y}_X, Z) \cong \lim_{\Delta} \mathsf{Map}_{\mathsf{Sch}}(X^\bullet/Y, Z).$$

Remark 0.1. The noetherian hypotheses are necessary.

Remark 0.2. We have to take the colimit over the Čech nerve in the category of derived schemes. This theorem fails if we try to do it in stacks: It is not true that any morphism $S \to \widehat{Y}_X$ factors locally for the étale topology through $X \to Y$.

Remark 0.3. Generalizing this to Artin stacks and derived schemes is nontrivial. We will see that this is already a nontrivial statement at the level of schemes.

Consider a smooth variety Y over a field k and a k-point y of Y. The nerve of y is simply

$$\mathsf{Spec}\, A^{\otimes n},$$

where $A = \mathrm{Sym}_k(y^*\Omega^1_Y)$. Functions on the colimit of this nerve is the limit of the cosimplicial object

$$[n] \mapsto \mathrm{Sym}\, y^*\Omega^1_Y[1],$$

which can be identified with $\widehat{\mathrm{Sym}(y^*\Omega^1_Y[1])}$.

Remark 0.4. We have to take the completion here, as the limit of this diagram lies in the second quadrant; hence, it involves a nonconverging spectral sequence forcing us to take the completion. We will see a similar phenomenon when we discuss the derived de Rham complex below.

Consider again a k-point of a scheme Y of finite type over k, and consider its derived based loop group

$$\Omega_y Y = \mathrm{pt} \times_Y \mathrm{pt}\,.$$

The above Theorem can be reformulated as

$$B(\Omega_y Y) \cong \operatorname{Spf} \widehat{\mathcal{O}_{Y,y}} = \widehat{Y}_y.$$

Remark 0.5. This is the algebro-geometric version of the fact that we can recover the connected component of a topological space Y containing a point $y \in Y$ as the classifying space of $\Omega_y Y$.

1 Preliminaries

1.1 Faithfully Flat Descent

1.1.1 Descent for Modules

Let $A \to B$ be a map of rings, and consider the cosimplicial A-algebra $A/B_\bullet$, the **Čech conerve of B under A**, given by

$$A/B_n = B^{\otimes_A n},$$

where the coface and codegeneracy maps are the obvious ones (insertion of the unit and multiplication of two factors). A B-module M gives rise to a cosimplicial $A/B_\bullet$-module $A/B_\bullet \otimes_B M$. We then have 🔗 023F

Lemma 1.1. Suppose that $f : A \to B$ has a section s. Then for any B-module M, we have a quasi-isomorphism $M \simeq A/B_\bullet \otimes_B M$, where we view both as cochain complexes via cosimplicial Dold-Kan 🔗 019H.

Proof. This follows immediately if we can show that the section induces a homotopy equivalence between the given complexes. Indeed, we have a homotopy equivalence $B \to A/B_\bullet$ of cosimplicial B-algebras. Since $fs = \mathrm{id}_A$, we only need to show that $sf \sim \mathrm{id}_B$. Define a homotopy $h_{n,0} = \mathrm{id}$, $h_{n,n+1} = (sf)^{n+1}$, and $h_{n,i} = \mathrm{id}_A^{n+1} \times (sf)^{n+1-i}$, where

$$h_n : B^{\times_A n} \times \mathrm{Hom}([n],[1]) \to B^{\times_A n}$$

viewed as a homotopy in the opposite category to Alg_B. Note that it is indeed a homotopy, see 🔗 019J Lemma 14.26.2. It then follows by formal nonsense that this defines a homotopy in Alg_B in the appropriate way. Thus, we have a homotopy equivalence $M \to M \otimes_B A/B_\bullet$ in the category of cosimplicial B-modules. (Cosimplicial) Dold-Kan preserves homotopy equivalences, so there is a corresponding one on the associated cochain complexes. Since the associated chain complex to the constant cosimplicial object M is just

$$M \overset{0}{\to} M \overset{1}{\to} M \overset{0}{\to} \cdots,$$

we are done. □

We now have

Proposition 1.2. Suppose that $f : A \to B$ is faithfully flat. Then for any B-module M we have a quasi-isomorphism on chain complexes $M \simeq A/B_\bullet \otimes_B M$.

Remark 1.1. A map of rings is **faithfully flat** if B is faithfully flat as an A-module. This in turn means that a short exact sequence of A-modules is exact if and only if its base change is exact.

Proof. Suppose we have a faithfully flat ring map $A \to A'$ such that the result holds for $A' \to B' = A' \otimes_A B$. It then follows that the result also holds for f. This is because $(M \otimes_A A') \otimes_{A'} A'/B'_\bullet \cong A' \otimes_A (M \otimes_A A/B_\bullet)$. Since $A \to A'$ is faithfully flat, exactness of the former complex implies exactness of the latter.

Moreover, we have such a faithfully flat map. Take $A' = B$ and notice that $B \to B' = B \otimes_A B$ has a section $s : b_1 \otimes b_2 \mapsto b_1 b_2$. Now, use the Lemma. □

1.1.2 Descent in the Simplicial Setting

Let $f : A \to B$ be a map of simplicial rings. Define the **derived Čech conerve of B under A**, $A/B_\bullet$, as the usual Čech conerve of the map $A \to P$ for any simplicial polynomial A-algebra resolution of B (for a canonical choice can take the standard resolution). Note that this is independent up to homotopy of the choice of P. Now, for any $M \in \mathsf{Ch}(A)$, define the **Adams completion of M along f** as

$$\mathrm{Comp}_A(M, f) := \mathrm{Tot}(M \otimes_A A/B_\bullet).$$

Remark 1.2. If $M = C$ for $C \in \mathsf{sAlg}_{A/}$, then $\mathrm{Comp}_A(C, f) \simeq \mathrm{Comp}_A(C, f \otimes_A \mathrm{id}_C)$ is naturally an $\mathbb{E}_\infty$-algebra.

We have the immediate analogs of the Lemma and Proposition above.

Lemma 1.3. Let $f : A \to B$ be a map of simplicial rings with a section. Then $\mathrm{Comp}_A(M, f) \simeq M$ for any $M \in \mathsf{Ch}(A)$.

Proposition 1.4. Let $f : A \to B$ be a faithfully flat map of simplicial rings. Then $\mathrm{Comp}_A(M, f) \simeq M$ for any $M \in \mathsf{Ch}(A)$.

Remark 1.3. Both are proved in the same way as in the case of ordinary rings.

1.2 Properties of the Adams Completion

Let $\mathscr{A}$ be a Grothendieck abelian category, and let $\mathbb{N}_0^{\mathrm{op}}$ be the category associated to the poset of non-positive integers. Then $\mathsf{Fun}(\mathbb{N}_0^{\mathrm{op}}, \mathscr{A})$ just denotes $\mathbb{N}_0^{\mathrm{op}}$-indexed diagrams in $\mathscr{A}$, and $D(\mathsf{Fun}(\mathbb{N}_0^{\mathrm{op}}, \mathscr{A}))$ is given by chain complexes of such diagrams localized at quasi-isomorphisms. Notice that objects $K \in D(\mathsf{Fun}(\mathbb{N}_0^{\mathrm{op}}, \mathscr{A}))$ can be given via a complete, separated filtration on $\widehat{K} = \lim K_k \in D(\mathscr{A})$, i.e. one such that $\widehat{K} = \lim \widehat{K}/F^i \widehat{K}$ and $\cap F^i \widehat{K} = 0$. Indeed, we can view the system of homotopy kernels $\ker(\widehat{K} \to K_k) \in D(\mathsf{Fun}(\mathbb{N}_0^{\mathrm{op}}, \mathscr{A}))$ as such a filtration on $\widehat{K}$. Lastly, notice that a cochain complex K over $\mathscr{A}$ defines an object $D \in D(\mathsf{Fun}(\mathbb{N}_0^{\mathrm{op}}, \mathscr{A}))$ via $K_k := K/\sigma^{\geq k} K$, where $\sigma^{\geq k}$ denotes the stupid filtration of K in cohomological degrees $\geq k$. Moreover, we have $K \simeq \lim K_k$, and we refer to the resulting filtration on K as the stupid filtration.

Now, given a Grothendieck abelian category $\mathscr{A}$, we say that an object $M \in \mathsf{Fun}(\mathbb{N}_0^{\mathrm{op}}, \mathscr{A})$ is **strict essentially 0** if $\exists k \in \mathbb{N}_0 | M_n \to M_m$ is 0 for any $n - m \geq k$. We say that an object $K \in D(\mathsf{Fun}(\mathbb{N}_0^{\mathrm{op}}, \mathscr{A})$ is **strict essentially 0** if the objects of $\mathsf{Fun}(\mathbb{N}_0^{\mathrm{op}}, \mathscr{A})$ defined by $\mathrm{H}^i(K)$ are strict essentially 0 for every i.

Remark 1.4. If $K \in D(\mathsf{Fun}(\mathbb{N}_0^{\mathrm{op}}, \mathscr{A}))$ is strict essentially 0, then $\lim_n K_n \simeq 0$. Indeed, the limit in $\mathbb{N}_0^{\mathrm{op}}$ is given by supremum; hence, the limit of each of the cohomology groups will be 0, so that $\widehat{K} \simeq 0$.

Remark 1.5. Notice that being strict essentially 0 is just the appropriate analog of the trivial Mittag-Leffler condition. This motivates the second lemma below.

Lemma 1.5. Let $N \subset M$ be finitely generated modules over a noetherian ring A, and let $\mathfrak{a} \subset A$ be an ideal. Consider the map

$$f : \{N/\mathfrak{a}^n N\} \to \{N/(\mathfrak{a}^n M \cap N)\}$$

in $\mathsf{Fun}(\mathbb{N}_0^{\mathrm{op}}, \mathsf{Mod}_A)$. Then f is surjective with kernel strict essentially 0.

Proof. Surjectivity is obvious. The latter assertion is given by the Artin-Rees lemma. □

Lemma 1.6 (Strict Essentially 0 Systems Form an Ideal)**.** Let A be a ring and $K \in D^{\leq 0}(\mathsf{Fun}(\mathbb{N}_0^{\mathrm{op}}, \mathscr{A}))$ a strict essentially 0 system, and let $M \in D^{\leq 0}(\mathsf{Fun}(\mathbb{N}_0^{\mathrm{op}}, \mathscr{A}))$ another system. Then $\{K_n \otimes_A M_n\}$ is strict essentially 0 with $\lim_n K_n \otimes_A M_n \simeq 0$.

Proof. Since the condition to be strict essentially 0 in particular gives us the Mittag-Leffler condition, we then have the following short exact sequence (Theorem 3.5.8 Weibel).

$$1 \to \lim_n \mathrm{H}^{-i}(K_n \otimes_A M_n) \to \mathrm{H}^{-i}(\lim_n (K_n \otimes_A M_n) \to \lim_n{}^1 \mathrm{H}^{-i-1}(K_n \otimes_A M_n) \to 1.$$

Since both $\lim$ and $\lim^1$ vanish for a strict essentially 0 system, it suffices to show that $\mathrm{H}^{-i}(K_n \otimes_A M_n)$ is a strict essentially 0 system for each $i \in \mathbb{N}_0$. But the Künneth spectral sequence gives a finite filtration on $\mathrm{H}^{-i}(K_n \otimes_A M_n)$ with graded pieces subquotients of $\mathrm{Tor}_j^A(\mathrm{H}^{-k}(K_n), M_n)$ for $j + k = i$ with $j \in \mathbb{N}_0$, $k \leq i$. □

Lemma 1.7 (Quillen)**.** Let $\mathfrak{a} \subset A$ be an ideal in a noetherian ring A, and let M be a finitely generated A-module. Then the cone of the map $\{M \otimes_A A/\mathfrak{a}^n\} \to \{M/\mathfrak{a}^n M\}$ of objects in $D(\mathsf{Fun}(\mathbb{N}_0^{\mathrm{op}}, \mathsf{Mod}_A))$ is strict essentially 0.

Proof. By Künneth, it suffices to check that $\{\mathrm{Tor}_i^A(M, A/\mathfrak{a}^n)\}$ is strict essentially 0 for $i \in \mathbb{N}$; moreover, we can just shift dimensions to see that we need only check this for $i = 1$. Write M as the quotient of a finite free A-module, $M = F/K$. We then have

$$\mathrm{Tor}_1^A(M, A/\mathfrak{a}^n) \cong \ker(K/\mathfrak{a}^n K \to F/\mathfrak{a}^n K) \cong (\mathfrak{a}^n F \cap K)/\mathfrak{a}^n K,$$

so the statement follows from Lemma 1.5. □

Proposition 1.8. Let $\mathfrak{a} \subset A$ be an ideal in a noetherian ring, and let M be a finitely generated A-module. For any $K \in D^{\leq 0}(\mathsf{Fun}(\mathbb{N}_0^{\mathrm{op}}, \mathsf{Mod}_A))$, the natural map induces an equivalence

$$\varphi : \lim(M \otimes A/\mathfrak{a}^n \otimes K_n) \xrightarrow{\cong} \lim(M/\mathfrak{a}^n M \otimes K_n).$$

Proof. Define $F : D(\mathsf{Fun}(\mathbb{N}_0^{\mathrm{op}}, \mathsf{Mod}_A)) \to D(\mathsf{Mod}_A)$ as the composition of $\lim$ with $\{- \otimes_A K_n\}$. Notice that φ is F applied to the natural map $\{M \otimes A/\mathfrak{a}^n\} \to \{M/\mathfrak{a}^n\}$. Now, use the Lemmas 1.6 and 1.7 above to finish the proof, noting that F is exact. □

Remark 1.6. Notice that the object $\{M \otimes_A A/\mathfrak{a}^n\}$ is independent of the choice of flat resolution of M used to compute it.

Lemma 1.9. Let $f : A \to B$ be a map of rings, and view a B-module M as an A-module via f. Then the map $M \to M \otimes_A A/B_\bullet$ is a homotopy equivalence of cosimplicial A-modules. In particular, $M \simeq \mathrm{Comp}_A(M, f)$.

Proof. The B-action on M defines the homotopy. ☐

Theorem 1.10 (Carlsson)**.** Let $\mathfrak{a} \subset A$ be an ideal in a noetherian ring A, and let M be a finitely generated A-module. There is a natural isomorphism $\widehat{M} \cong \mathrm{Comp}_A(M, \mathfrak{a})$, where $\mathrm{Comp}_A(M, \mathfrak{a})$ denotes the Adams completion of M with respect to $A \to A/\mathfrak{a}$.

Proof. Let $F \in \mathrm{End}(D(\mathsf{Mod}_A))$ be the exact functor $M \mapsto \mathrm{Tot}(M \otimes_A A/(A/\mathfrak{a})_\bullet$. We have an obvious natural transformation $\eta : \mathrm{id} \to F$, and we claim that $\eta_M : M \to F(M)$ is an equivalence whenever M is an $A/\mathfrak{a}^n$-module for any $n \in \mathbb{N}$. Since both id and F are exact, we can apply them to the short exact sequence

$$0 \to \mathfrak{a}^k \to A/\mathfrak{a}^n \to A/\mathfrak{a}^k \to 0$$

for $k < n$ to reduce to the case $n = 1$ (i.e. we use a dévissage argument). But this case is handled by Lemma 1.9. Thus, for any $n \in \mathbb{N}_0$ and finitely generated A-module M we have equivalences

$$\eta_{M/\mathfrak{a}^n M} : M/\mathfrak{a}^n M \xrightarrow{\simeq} \mathrm{Tot}(M/\mathfrak{a}^n M \otimes_A A/(A/\mathfrak{a})_\bullet).$$

Take a limit over $\mathbb{N}_0^{\mathrm{op}}$ to obtain the equivalence

$$\hat{\eta} : \widehat{M} = \lim M/\mathfrak{a}^n M \xrightarrow{\simeq} \mathrm{Tot}(\lim(M/\mathfrak{a}^n M \otimes_A A/(A/\mathfrak{a})_\bullet)).$$

We have a natural map $\mathrm{Tot}(M \otimes_A A/(A/\mathfrak{a})_\bullet) \to \mathrm{Tot}(\lim(M/\mathfrak{a}^n M \otimes_A A/(A/\mathfrak{a})_\bullet)$, so we need only check that $M \otimes_A A/(A/\mathfrak{a})_\bullet$ and $\lim(M/\mathfrak{a}^n M \otimes_A A/(A/\mathfrak{a})_\bullet)$ are equivalent in $D(\mathsf{Fun}(\Delta, \mathsf{Mod}_A))$ under the natural map

$$\varphi : M \otimes_A A/(A/\mathfrak{a})_\bullet \to \lim(M/\mathfrak{a}^n M \otimes_A A/(A/\mathfrak{a})_\bullet).$$

Now, the term at level $[m] \in \Delta$ in the source is $M \otimes_A (A/\mathfrak{a})^{\otimes(m+1)}$, while in the target it is $\lim(M/\mathfrak{a}^n M \otimes (A/\mathfrak{a})^{\otimes(m+1)})$. We can then apply Proposition 1.8 three times:

$$\begin{aligned}
\lim(M/\mathfrak{a}^n M \otimes_A (A/\mathfrak{a})^{\otimes(m+1)}) &= \lim(M \otimes A/\mathfrak{a}^n \otimes (A/\mathfrak{a})^{\otimes(m+1)}) \\
&= \lim(M \otimes A/\mathfrak{a}^n \otimes A/\mathfrak{a} \otimes (A/\mathfrak{a})^{\otimes m}) \\
&= \lim(M \otimes (A/\mathfrak{a})/\mathfrak{a}^n \otimes (A/\mathfrak{a})^{\otimes m}) \\
&= \lim(M \otimes (A/\mathfrak{a})^{\otimes(m+1)}) \\
&= M \otimes (A/\mathfrak{a})^{\otimes(m+1)}.
\end{aligned}$$

Since this is exactly the source of φ, we're done. ☐

2 Algebraic and Derived De Rham Cohomology

2.1 The Hodge Filtration

Let X be a scheme over S, and consider the de Rham complex $\Omega^*_{X/S}$. The **Hodge-to-de-Rham spectral sequence** is given by

$$E_1^{p,q} = \mathrm{H}^q(X, \Omega^p_{X/S}),$$

which are exactly the Hodge cohomology groups of X over S. The differential $d_1^{p,q}$ is induced by the usual differential $d : \Omega^p_{X/S} \to \Omega^{p+1}_{X/S}$. We call the filtration on $\mathrm{H}^n(X/S) := \mathrm{H}^n(R\Gamma(X, \Omega^*_{X/S}))$ induced by this spectral sequence the **Hodge filtration**. It is explicitly given by

$$F^p\mathrm{H}^n(X/S) = \mathrm{im}\left(\mathrm{H}^n(X, \sigma^{\leq p}\Omega^*_{X/S}) \to \mathrm{H}^n(X/S)\right),$$

where $\sigma^{\leq p}$ is the stupid truncation.

2.2 Algebraic de Rham Cohomology

Let $f : A \to B$ be a finite type map of noetherian $\mathbb{Q}$-algebras, and fix a presentation $F \to B$ with F a finite type polynomial A-algebra. Define the **algebraic de Rham complex** $\Omega^H_{B/A} \in D(\mathsf{Mod}_A)$ as

$$\Omega^H_{B/A} := \Omega^*_{F/A} \otimes_F \widehat{F},$$

where $\widehat{F}$ is the completion of F along $F \to A$, i.e. using $\ker(F \to A) = I$. Note that this construction is independent of the choice of F. Further, we have two filtrations on $\Omega^H_{B/A}$: The filtration defined by the Hodge filtration on $\Omega^*_{F/A}$ is called the **formal Hodge filtration** (it depends on F); the one obtained by tensoring the I-adic filtration on $\widehat{F}$ with the Hodge filtration on $\Omega^*_{F/A}$ is called the **infinitesimal Hodge filtration** (it is independent of F). Denote the latter filtration by $\mathrm{Fil}^*_{\mathrm{inf}}$. It is explicitly defined by

$$\Omega^H_{B/A}/\mathrm{Fil}^p_{\mathrm{inf}} := \left(F/I^p \to F/I^{p-1} \otimes_F \Omega^1_{F/A} \to F/I^{p-2} \otimes_F \Omega^2_{F/A} \to \cdots\right)$$

where we set $I^k = F$ for $k \leq 0$.

Example 2.1. Let $A = \mathbb{C}[x,y]/(y^2 - x^3)$, take $F = \mathbb{C}[x,y]$, so that

$$\Omega^H_{A/\mathbb{C}} \simeq \left(\widehat{F} \to \widehat{F}dx \oplus \widehat{F}dy \to \widehat{F}dx \wedge dy\right),$$

where $\widehat{F}$ is the completion of F along $(y^2 - x^3)$. Then $\mathrm{Spec}(A)^{\mathrm{an}}$ is contractible, so $R\Gamma(\mathrm{Spec}(A)^{\mathrm{an}}, \mathbb{C}) \simeq \mathbb{C}$. Now, notice that

$$\Omega^H_{A/\mathbb{C}} \simeq \mathbb{C}.$$

Remark 2.1. The general theorem of Hartshorne is that for a finite-type $\mathbb{C}$-algebra A, $\Omega^H_{A/\mathbb{C}}$ computes the Betti cohomology of $\mathrm{Spec}(A)^{\mathrm{an}}$.

2.3 Derived de Rham Cohomology

2.3.1 The Derived De Rham Complex

Let $A \to B$ be a map of rings. Resolve B by a polynomial A-algebra (say the standard resolution). Note that $\Omega^*_{P/A}$ is a simplicial, dg A-algebra, where the i^{th} column is the de Rham complex $\Omega^*_{P_i/A}$ and the j^{th} row is the P-module $\Omega^j_{P/A}$. Since $\Omega^j_{P/A}$ is a flat P-module for $j = 1$, it is flat for any j. Since $P \to B$ is a quasi-isomorphism, we thus have the adjunction arrow

$$\Omega^j_{P/A} \to \Omega^j_{P/A} \otimes_P B = \wedge^j \mathbb{L}_{B/A}, \tag{2.1}$$

which is also a quasi-isomorphism for all j. The total complex of $\Omega^*_{P/A}$ is denoted

$$\mathbb{L}\Omega^*_{B/A}$$

and called the **derived de Rham complex of** B/A. We can globalize this for a morphism $f : X \to S$ of schemes by setting

$$\mathbb{L}^*_{X/S} = \mathbb{L}\Omega^*_{\mathcal{O}_X/f^{-1}\mathcal{O}_S}.$$

This complex is equipped with the **Hodge filtration**:

$$F^k\mathbb{L}\Omega^*_{B/A} := \mathrm{Tot}\left(\cdots \to 0 \to \Omega^k_{P/A} \to \Omega^{k+1}_{P/A} \to \cdots\right).$$

It has associated graded

$$\mathrm{gr}\,\mathbb{L}\Omega^*_{B/A} \xrightarrow{\simeq} \wedge\mathbb{L}_{B/A}[-*] := \bigoplus_i (\wedge^i \mathbb{L}_{B/A})[-i].$$

Indeed, the adjunction arrows (2.1) give us the desired graded quasi-isomorphisms.

Remark 2.2. We can use any free resolution $P \to B$ instead of the standard one without changing the above quasi-isomorphism. Indeed, since $\mathbb{L}_{B/A}$ is flat, the second term is by definition $\mathbb{L} \wedge \mathbb{L}_{B/A}[-*]$. The invariance of $\Omega^1_{P/A}$ and $\mathbb{L}_{B/A}$ under choice of resolution gives the corresponding invariance of $\mathbb{L}\Omega^*_{B/A}$.

2.3.2 The Completed Derived De Rham Complex

Since $\Omega^*_{P/A}$ is in the second quadrant, the Hodge-to-de-Rham spectral sequence does not in general degenerate. It is thus convenient to work with the "completed" version. Define the **completed derived de Rham complex** as

$$\widehat{\mathbb{L}\Omega^*_{B/A}} := \lim \mathbb{L}\Omega^*_{B/A}/F^n\mathbb{L}\Omega^*_{B/A}.$$

2.3.3 The Simplicial Derived De Rham Complex and its Completion

We can upgrade the preceding discussion to a map of simplicial rings. Let $A \to B$ be a map of simplicial rings, and let $P \to B$ be a polynomial A-algebra resolution of B. Define the **Hodge-completed derived de Rham complex** $\widehat{\mathrm{dR}}_{B/A}$ of $A \to B$ as the completion of $|\Omega^*_{P/A}|$ for its Hodge filtration, i.e. $\widehat{\mathrm{dR}}_{B/A} =$

$\lim_{\mathbb{N}_0^{\mathrm{op}}} \widehat{\mathrm{dR}}_{B/A} / \mathrm{Fil}_H^k$, where $\widehat{\mathrm{dR}}_{B/A} / \mathrm{Fil}_H^k \in \mathsf{Ch}(A)$ is the totalization of the simplicial cochain complex $[n] \mapsto \sigma^{\leq k} \Omega^n_{P/A}$. As above, its graded pieces are computed by

$$\mathrm{gr}_H^k(\widehat{\mathrm{dR}}_{B/A}) \simeq \wedge^k \mathbb{L}_{B/A}[-k],$$

and they're independent of the choice of P. This also applies to any map of simplicial commutative rings in a topos.

Example 2.2. If $A \to B$ is smooth, then $\mathbb{L}_{B/A} \simeq \Omega^1_{B/A}$, so $\widehat{\mathrm{dR}}_{B/A}$ is the usual de Rham complex.

Example 2.3. Let A be a $\mathbb{Q}$-algebra, and let $B = A/(f)$, where $f \in A$ is regular. Then $L_{B/A} \simeq (f)/(f^2)[1]$. In particular, $\widehat{\mathrm{dR}}_{B/A} / \mathrm{Fil}_H^2$ is an extension of B by $\mathbb{L}_{B/A}[-1] = (f)/(f^2)$. The natural map $A \to \widehat{\mathrm{dR}}_{B/A}$ induces an equivalence $A/(f^2) \cong \widehat{\mathrm{dR}}_{B/A} / \mathrm{Fil}_H^2$. Moreover, this map induces equivalences $A/(f^n) \cong \widehat{\mathrm{dR}}_{B/A} / \mathrm{Fil}_H^n$ for each n, hence an equivalence $\widehat{A} \cong \widehat{\mathrm{dR}}_{B/A}$. By Künneth, this extends to an equivalence $A/\mathfrak{a}^n \cong \widehat{\mathrm{dR}}_{B/A} / \mathrm{Fil}_H^n$ for $B = A/\mathfrak{a}$ with $\mathfrak{a}$ any regular ideal.

Proof. Consider first the $\mathbb{Q}$-algebra map $\mathbb{Q}[x] \xrightarrow{1 \mapsto 1} \mathbb{Q}$. The transitivity triangle degenerates in this case, since $\mathbb{L}_{\mathbb{Q}/\mathbb{Q}} = 0$, so that

$$\begin{aligned} \mathbb{L}_{\mathbb{Q}/\mathbb{Q}[x]} &\cong \mathbb{L}_{\mathbb{Q}[x]} \otimes_{\mathbb{Q}[x]} \mathbb{Q}[1] \\ &\cong (x)/(x^2)[1]. \end{aligned}$$

Now, consider the $\mathbb{Q}$-algebra map $\mathbb{Q}[x] \xrightarrow{x \mapsto f} A$. By base change,

$$\mathbb{L}_{B/A} = \mathbb{L}_{\mathbb{Q}/\mathbb{Q}[x]} \otimes^{\mathbb{L}}_{\mathbb{Q}} B \cong (f)/(f^2)[1]$$

by the above.

Now, since $\mathbb{L}_{B/A} \cong (f)/(f^2)[1]$, we can use the fact that

$$\mathbb{L}_{B/A} = \Omega^1_{P/A} \otimes_P B$$

for $P \to B$ the standard free resolution to compute $\Omega^1_{P/A}$: It is given by $\mathbb{L}_{B/A}$ itself. Indeed, the standard resolution has i^{th} term $B^{\otimes_A i+1} = B$. Hence,

$$\Omega^1_{P/A} \otimes_P B = \Omega^1_{P/A}.$$

But notice that the map $P \to \mathbb{L}_{B/A}$ is given by the differential $P_1 \to (f)/(f^2)$ which is clearly the 0 map. Hence, we have a complex

$$B \xrightarrow{0} \mathbb{L}_{B/A} \simeq B \oplus (f)/(f^2) = A/(f^2).$$

Induction immediately gives the equivalences $A/(f^n) \cong \widehat{\mathrm{dR}}_{B/A} / \mathrm{Fil}_H^n$.

If $B = A/\mathfrak{a}$ with $\mathfrak{a} = (f_1, \ldots, f_n)$ regular, then the calculation above generalizes to

$$\mathbb{L}_{B/A} \cong \mathfrak{a}/\mathfrak{a}^2[1].$$

First, we use the $\mathbb{Q}$-algebra map $\mathbb{Q}[x_1,\dots,x_n] \xrightarrow{1\mapsto 1} \mathbb{Q}$. Next, we can use the $\mathbb{Q}$-algebra map $\mathbb{Q}[x_1,\dots,x_n] \xrightarrow{x_i\mapsto f_i} A$. The Koszul complex then gives a free resolution of B, so we get that

$$B = A \otimes^{\mathbb{L}}_{\mathbb{Q}[x_1,\dots,x_n]} \mathbb{Q},$$

which in turn gives

$$\mathbb{L}_{B/A} = \mathbb{L}_{\mathbb{Q}/\mathbb{Q}[x_1,\dots,x_n]} \otimes^{\mathbb{L}}_{\mathbb{Q}} B.$$

Now, our sequence becomes

$$B \to \mathfrak{a}/\mathfrak{a}^2[1] \to \wedge^2_B \mathfrak{a}/\mathfrak{a}^2[1] \to \cdots \to \wedge^n_B \mathfrak{a}/\mathfrak{a}^2[1].$$

Now, use the Künneth spectral sequence to prove the claim by induction. Namely, use that $K\left(f_1, f_2\right) = K(f_1) \otimes K(f_2)$ along with Künneth, where $K(\cdot)$ denotes the Koszul complex associated to the regular sequence $(\cdot)$. □

3 The Main Theorem

Proposition 3.1 (Quillen)**.** Let A be a simplicial ring, and let $\mathfrak{a} \subset A$ be a simplicial ideal with $\pi_0(\mathfrak{a}) = 0$ and such that $\mathfrak{a}_n \subset A_n$ is regular for each n. Then A is $\mathfrak{a}$-adically complete, i.e. $A \cong \lim A/\mathfrak{a}^n$ in the category of simplicial A-algebras.

Corollary 3.2. Let A be a simplicial $\mathbb{Q}$-algebra, and let $\mathfrak{a} \subset A$ an ideal with $\pi_0(\mathfrak{a}) = 0$. Then A admits a functorial, complete, separated $\mathbb{N}_0^{\mathrm{op}}$-indexed filtration Fil^k_H whose associated $\mathbb{N}_0^{\mathrm{op}}$-indexed system of quotients is

$$\{A/\mathrm{Fil}^k_H\} \simeq \{\widehat{\mathrm{dR}}_{(A/\mathfrak{a})/A}/\mathrm{Fil}^k_H\}.$$

In particular, $A \simeq \widehat{\mathrm{dR}}_{(A/\mathfrak{a})/A}$.

Proof. One can define a simplicial model structure on the simplicial category of pairs $(A, \mathfrak{a})$ comprising a simplicial algebra A together with simplicial ideals $\mathfrak{a} \subset A$ as follows. A map $(A, \mathfrak{a}) \to (B, \mathfrak{b})$ is a (trivial) fibration if and only if $A \to B$, $\mathfrak{a} \to \mathfrak{b}$ are so as maps of simplicial sets. The cofibrant objects are pairs $(F, \mathfrak{f})$ with each F_n a polynomial algebra on a set $\{x_n\}$ of generators, and each $\mathfrak{f}_n \subset F_n$ an ideal defined by a subset $\{y_n\} \subset \{x_n\}$ of the polynomial generators such that both the x_n and y_n are preserved by the degeneracies. In particular, for each cofibrant $(F, \mathfrak{f})$, the ideal $\mathfrak{f}$ is termwise regular. Now the claim follows from the above Proposition by cofibrant replacement. □

Theorem 3.3. Let $f : A \to B$ be a surjection of simplicial $\mathbb{Q}$-algebras. Then $\mathrm{Comp}_A(A, f) \in \mathsf{Ch}(A)$ admits a canonical $\mathbb{N}_0^{\mathrm{op}}$-indexed, separated, complete filtration Fil^*_H whose associated $\mathbb{N}_0^{\mathrm{op}}$-indexed system of quotients is

$$\{\mathrm{Comp}_A(A, f)/\mathrm{Fil}^k_H\} \simeq \{\widehat{\mathrm{dR}}_{B/A}/\mathrm{Fil}^k_H\}.$$

In particular, $\mathrm{Comp}_A(A, f) \simeq \widehat{\mathrm{dR}}_{B/A}$.

Proof. We can assume that B is A-cofibrant, i.e. B is a simplicial polynomial A-algebra. Since $A \to B$ is surjective, $\pi_0(B_m) \simeq \pi_0(B)$; hence, by Corollary 3.2, we have

$$\{B_m / \mathrm{Fil}_H^k\} \simeq \{\widehat{\mathrm{dR}}_{B/B_m} / \mathrm{Fil}_H^k\}.$$

Moreover, by functoriality, we get identifications of $\mathbb{N}_0^{\mathrm{op}}$-indexed systems of cosimplicial $A/B_\bullet$-complexes

$$\{(A/B_\bullet)/\mathrm{Fil}_H^k\} \simeq \{\widehat{\mathrm{dR}}_{B/(A/B_\bullet)} / \mathrm{Fil}_H^k\}.$$

Limits commute with limits, so we have an identification

$$\{\mathrm{Comp}_A(A, f)/\mathrm{Fil}_H^k\} \simeq \{\mathrm{Tot}(\widehat{\mathrm{dR}}_{B/(A/B_\bullet)} / \mathrm{Fil}_H^k)\}.$$

But we also have a map by functoriality

$$\varepsilon : \{\widehat{\mathrm{dR}}_{B/A/B} / \mathrm{Fil}_H^k\} \to \{\mathrm{Tot}(\widehat{\mathrm{dR}}_{B/(A/B_\bullet)} / \mathrm{Fil}_H^k)\},$$

so we need only check that ε is an equivalence. Passing to the associated graded, we find

$$\mathrm{gr}^k(\varepsilon) : \wedge^k \mathbb{L}_{B/A}[-k] \to \mathrm{Tot}(\wedge^k \mathbb{L}_{B/(A/B_\bullet)}[-k])$$

which is an equivalence by the following two lemmas. □

Lemma 3.4. Let A be a cosimplicial simplicial ring. The forgetful functor $\mathsf{csMod}_A \to \mathsf{sMod}_{A_0}$ has a left adjoint F; for any $M \in \mathsf{sMod}_{A_0}$, $F(M)$ is homotopy equivalent to 0 in csMod_A.

Remark 3.1. The proof is formal nonsense.

Lemma 3.5. Let $A \to B$ be a map of simplicial rings. Then

$$\mathrm{Tot}(\wedge^k \mathbb{L}_{B/(A/B_\bullet)}) \simeq \wedge^k \mathbb{L}_{B/A}.$$

Proof. Choose a simplicial polynomial A-algebra resolution $P \to B$, and let $\mathfrak{a} \subset A/P_\bullet$ be the cosimplicial simplicial ideal defining the augmentation $A/P_\bullet \to P$. Then

$$\wedge^k \mathbb{L}_{B/(A/B_\bullet)} = \wedge^k (I/I^2[1]) \in \mathsf{cs\,Mod}_P.$$

By the Lemma, $\Omega^1_{(A/P_\bullet)/A} \otimes_{A/P_\bullet} P$ and its wedge powers are homotopy equivalent to 0 in csMod_P. Thus, take $\wedge^k$ of the cosimplicial transitivity triangle

$$\Omega^1_{(A/P_\bullet)/A} \otimes_{A/P_\bullet} P \to \Omega^1_{P/A} \to I/I^2[1]$$

of simplicial P-modules. □

We can now prove the main theorem.

Theorem 3.6 (Main Theorem)**.** Let $A \to B$ be a finite type map of noetherian $\mathbb{Q}$-algebras. Then there is a filtered A-algebra

$$\widehat{\mathrm{dR}}_{B/A} \to \Omega^H_{B/A}$$

map which is an equivalence of the underlying algebras.

Proof. If we fix a finite type polynomial A-algebra F with a surjection of A-algebras $F \to B$, then $\Omega^H_{B/A} \simeq \Omega^*_{F/A} \otimes_F \widehat{F}$. It is a fact that

$$\Omega^H_{B/A} \simeq \mathrm{Tot}\big(\widehat{F} \rightrightarrows \widehat{F \otimes_A F} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \widehat{F \otimes_A F \otimes_A F} \cdots\big),$$

where the completion is along the composition $F^{\otimes_A n} \to F \to B$. By Carlsson's Theorem (1.10), we have

$$\Omega^H_{B/A} \simeq \mathrm{Tot}(\mathrm{Comp}_A((A/F_\bullet) \to B).$$

Now, use Theorem 3.3 applied to the maps $F^{\otimes_A n} \to B$ to obtain

$$\Omega^H_{B/A} \simeq \mathrm{Tot}\left(\widehat{\mathrm{dR}}_{B/(A/F_\bullet)}\right).$$

We see that $\Omega^H_{B/A}$ thus has a complete separated $\mathbb{N}_0^{\mathrm{op}}$-indexed filtration $\mathrm{Fil}^*_{H'}$ such that

$$\{\Omega^H_{B/A} / \mathrm{Fil}^k_{H'}\} \simeq \left\{\mathrm{Tot}\left(\widehat{\mathrm{dR}}_{B/(A/F_\bullet)} / \mathrm{Fil}^k_H\right)\right\}.$$

Now, we can use the argument at the end of the proof of Theorem 3.3. □