Cohomological hall aglebra as BPS states

Summary:

- Introduce: What are D-branes for type B string theory?
- What are BPS string states
- Describe donaldson thomas invariants and their count of BPS states
- example: exceptional collection of sheaves gives you a quiver
- Cohomological hall algebra counts these bps states

Talk

The last two weeks, we learned about the definition of a Cohomological hall algebra. Konstevich and Soibelman introduced this algebra as a mathematically precise model of a BPS algebra. The relationship between CoHA and physics is buried behind several classical constructions, and somewhat hard to piece out. Today I want to trace through these, and explain what a CoHA has to do with physics.

Part 1: Supersymmetry and BPS states

Following Neitzke, Lectures on BPS states and spectral networks

We are interested in supersymmetric quantum field theories. Let's take a Atiyah-like definition of a quantum field theory.
Definition: A *Quantum field theory* in d dimensions is a functor from d-cobordims to the category of hilbert spaces. Explicitly,

- To each $d-1$ manifold with riemannian metric (X, g), we assign a hilbert space $Z(X, g)$ with inner product.
- To each d dimensional cobordism (W, g) with $\partial W = X_0 \coprod X_1$, we assign a unitary map $Z(W, g) : Z(X_0, g) \to Z(X_1, g)$.

Remarks:

- our cobordisms come with a metric. We have not yet specalized to topological quantum field theories.
- The hilbert spaces may be infinite dimensional, in fact they often are.
- We will usually be interested in quantum field theory on Euclidean space. We will denote the hilbert space $\mathcal{H} = Z(\mathbb{R}^{d-1})$.

Now we impose supersymmetry. We fix a "supersymmetry algebra", a Lie superalgebra ($\mathbb{Z}_2$-graded Lie algebra) $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$. In particular, for a dimension d quantum field theory, we take $\mathcal{A}$ to have even part equal to the poincare lie algebra, $\mathcal{A}^0 = \mathfrak{so}(d-1, 1)$.
Definition: A *Supersymmetric Quantum field theory* is a d dimensional quantum field theory, with an action of $\mathcal{A}$ on each hilbert space $\mathcal{H}_X$. In particular, each hilbert space $\mathcal{H}$ carries a $\mathbb{Z}_2$ grading $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ with a representation of $\mathcal{A}$, such that $\mathcal{A}^0$ preserves the grading and $\mathcal{A}^1$ swaps the grading.

We can decompose $\mathcal{H}$ into irreducible representations of $\mathcal{A}$. The BPS states are a special class of irriducible representation:
Definition: a *BPS state* of $\mathcal{H}$ is a vector $v \in \mathcal{H}$ belonging to a irreducible representation of $\mathcal{A}$ where a nontrivial subspace of $\mathcal{A}_1$ acts trivially.

Example: supersymmetric quantum mechanics

To see BPS states in action, let us look a supersymmetric 1D quantum field theory. This will give us a supersymmetric model of quantum mechanics. 1D quantum field theories assign the 0 dimensional manifold to a hilbert space $\mathcal{H} = Z(\mathrm{pt})$. Now we define the supersymmetry algebra which will act on this hilbert space:

Definition: $d=1, \mathcal{N}=2$ supersymmetry algebra:

$$\mathcal{A}=\mathcal{A}^0 \oplus \mathcal{A}^1 \qquad \mathcal{A}^0=\mathbb{C}\langle H\rangle \qquad \mathcal{A}^1=\mathbb{C}\langle Q, \bar{Q}\rangle$$

The generators $H, Q, \bar{Q}$ have a single nontrivial commutation relation:

$$[Q, \bar{Q}]=Q \bar{Q}+\bar{Q} Q=2 H$$

A unitary representation of $\mathcal{A}$ is a hilbert space $\mathcal{H}=\mathcal{H}^0 \oplus \mathcal{H}^1$, where H acts by a self-adjoint operator preserving grading, and Q and $\bar{Q}$ are adjoint to eachother, and exchange the grading.

Remark: Notice that $[Q, Q]=Q^2=0$. Oftentimes, $\mathcal{H}$ carries a $\mathbb{Z}$ grading, and $\mathcal{H}^0$ and $\mathcal{H}^1$ come from the even and odd parts of the $\mathbb{Z}$ grading, and Q is an operator on $\mathcal{H}$ of degree 1. Then, Q endows $\mathcal{H}$ with a differential graded structure. As far as i know, this is why supersymmetry has something to say about derived geometry.

Example 1D QFT: 1D SUSY sigma model Our main example of a 1D quantum field theory comes from the supersymmetric quantum theory of a particle on a riemannian manifold M. This has the hilbert space consisting of differential forms on M, with its standard L^2 inner product. The $\mathbb{Z}_2$ grading comes from the degree of the form

$$\mathcal{H}=\Omega^*(M) \qquad \mathcal{H}^0=\bigoplus_k \Omega^{2 k}(M) \qquad \mathcal{H}^1=\bigoplus_k \Omega^{2 k+1}(M)$$

The representation of $\mathcal{A}$ is defined by

$$Q=\mathrm{d}: \Omega^{2 k} \rightarrow \Omega^{2 k+1} \qquad \bar{Q}=d^*: \Omega^{2 k} \rightarrow \Omega^{2 k-1} \qquad H=\frac{1}{2} \Delta: \Omega^{2 k} \rightarrow \Omega^{2 k}$$

Representation theory of $\mathcal{A}$: We move back to the abstract setting. Suppose V is an irriducible unitary represetnation of $\mathcal{A}$. Since H is in the center of $\mathcal{A}$, it must act on V by a scalar E. We call E the "energy" of V. We have a lower bound on the energy: Observe that H acts by a positive operator:

$$\begin{aligned}
\langle\psi, H \psi\rangle & =\langle\psi,(Q \bar{Q}+\bar{Q} Q) \psi\rangle \\
& =\langle\psi, Q \bar{Q} \psi\rangle+\langle\psi, \bar{Q} Q \psi\rangle \\
& =\langle Q \psi, Q \psi\rangle+\langle\bar{Q} \psi, \bar{Q} \psi\rangle \\
& =||Q \psi||^2+||\bar{Q} \psi||^2 \geq 0
\end{aligned}$$

Thus $E \geq 0$. The irriducible representations V behave very differently when they saturate this bound. Let's go through both cases:

- $E>0$:
  - irriducible representations are 2 dimensional (draw flowchart here)
- $E=0$:
  - irreducible representations are one dimensional. Indeed, choose any vector $v \in \operatorname{ker} H$. From the above computation, we see

    $$\langle v, H v\rangle=||Q v||^2+||\bar{Q} v||^2=0 \implies Q v=\bar{Q} v=0$$

    So the $\mathcal{A}$-representation generated by v is the one dimensional subspace spanned by v, and $\mathcal{A}$ acts trivially.
  - We call states with $E=0$ **BPS states**

BPS states of SUSY Quantum mechanics
A form $\psi \in \Omega^*(M)$ is BPS if $H \psi=\frac{1}{2} \Delta \psi=0$. That is, BPS states are harmonic forms on M. By the hodge theorem, the vector space of BPS states $\mathcal{H}^{BPS}$ is isomorphic to the cohomology $H^*(M)$.
Remarks:

- Even though our quantum field theory depends on the Riemannian structure of M, the BPS states are *invariant under deformation of the Riemannian metric*. BPS states measure topological information about the theory.
- $\mathcal{H}^{BPS}$ is isomorphic to the De Rahm cohomology $H^*(M)$. In our setup, this is the cohomology of the action of Q on $\mathcal{H}$.

For a topological theory, "Physical states" are the cohomology of the action of Q. We call these BRST cohomology

BPS states for higher dimensional supersymmetry algebras

The universal enveloping algebra of the supersymmetry algebra has a center, which acts by scalars on any irriducible representation V. We classify V by these scalars. There are two generators of the center:

- The casimir of the even part, $\mathfrak{so}(d-1,1)$. This acts by a real scalar $E(V)$, hich we call the "energy"
- By bracketing together different odd elements, we get acentral element Z. This acts by the scalar $Z(V) \in \mathbb{C}$, which we call the "central charge"
  A similar computation to above gives a bound for the energy in terms of the central charge, which we call the BPS bound:

  $$E(V) \geq |Z(V)|$$

  We call an irrep V *Short* if it saturates the BPS bound, and *long* otherwise. We can (schematically) decompose $\mathcal{H}$ into short and long irreps:

  $$\mathcal{H} = \bigoplus_{\text{short}} V \oplus \bigoplus_{\text{long}} V$$

  Definition a *BPS state* is a vector $v \in \mathcal{H}$ belonging to a short irrep.

Facts about BPS states:

- Things saturating the BPS bound preserve half the supersymmetry. That is, if V is a short irrep, there is a half dimensional subspace of $\mathcal{A}^1$ whose action on V vanishes.
- Short irreps are lower dimensional than long irreps (hence the name short and long)
- This means short irreps survive under deformation! A single short irrep cannot deform to become long, unless it combines with other short irreps. In pariticular, *A signed count of BPS states is deformation invariant*.

This last fact is very significant. It means we can use BPS states to probe the nonperturbitive behievor of our theory. Start with the theory in a limit where it is easier to understand. This is usually the low energy limit (for $N=2$ quantum mechanics, we scale the metric on M to be very large). Then, deform to the theory to what we actually want, and we can be sure of the existence of some BPS states. This technique came into promenence when Seiberg and Witten used it to fully solve $\mathrm{d}=4, \mathcal{N}=2$ super yang mills theory. See Monopole Condensation, And Confinement In N=2 Supersymmetric Yang-Mills Theory.

Part 2: BPS states in String theory

(This part follows section 5.4.2 of D-branes and Mirror symmetry)

We wish to define interesting quantum field theories on $\mathbb{R}^{3,1}$. To do this, we upgrade to a 10 dimensional manifold using a Calabi-yau 3-fold X.

$$M = \mathbb{R}^{3,1} \times X$$

Then we study a relativly simple field theory on M and "integrate out" the X coordinates. The complexities on $\mathbb{R}^{3,1}$ arise from the complexities of the geometry and topology of X.

In order to probe the geometry of X, we will use string theory. This is a 2 dimensional quantum field theory, studying maps of a 2 dimensional surface Σ into M. Schematically, the quantum field theory is defined by an action functional $S : \mathrm{Maps}(\Sigma, M) \to \mathbb{R}$ via the partition function

$$Z(M) = \int_{\mathrm{Maps}(\Sigma, M)} e^{iS(u)} \mathrm{d}u$$

We will turn this into an "effective" 4 dimensional theory, by introducing boundary conditions to our 2D theory.

Here's how to turn a boundary condition into a "particle". Fix a sub-manifold L of dimension p in X. In the buisness, we call this a p-brane. A "particle" in $\mathbb{R}^{3,1}$ is something which traces out a timelike path $c \cong \mathbb{R}^{0,1} \subset \mathbb{R}^{3,1}$. We represent this geometrically. as a submanifold

$$c \times D \subset \mathbb{R}^{3,1} \times X$$

The submanifold D represents the internal structure of the particle. Two particles $c \times D$ and $c' \times D'$ interact via string theory. In particular, we demand our 2 dimensional surface Σ is the open strip, with boundary components $\partial\Sigma = R^0 \sqcup R^1$. We impose that one boundary line lives on one submanifold, and the other boundary line lives on the other. Define the moduli space of maps with these boundary conditions:

$$\mathcal{M}(L, L') = \{u : \Sigma \to M |\, u|_{R^0} \subset c \times D \quad u|_{R^1} \subset c' \times D'\}$$

Schematically, our partition function for the resulting 4D theory is the integral over maps with these boundary conditions

$$Z(\mathbb{R}^{3,1}, L, L') = \int_{\mathcal{M}(L,L')} e^{iS(u)}$$

In other words, *Interactions between particles are governed by integrating over morphsims between branes in X.*

The details depend on our model of string theory. We will use type IIA string theory. The properties of this particle depends on the geometry of X, which will be a kahler structure with metric g and kahler form ω. The mass of a particle associated to a brane D, which we denote by $M(D)$, is given by the riemannian volume of D. For a p dimensional brane (p must be even), the central charge $Z(D)$ is given by

$$Z(D) = \int_D \frac{\omega^{p/2}}{(p/2)!}$$

this is the symplectic volume of the kahler form, pulled back to D. Notice that the riemannian volume is always greater than the kahler volume, and we have the BPS bound

$$M(D) = \mathrm{vol}_g(D) \geq \int \frac{\omega^{p/2}}{(p/2)!} = Z(D)$$

A BPS particle corresponds to a brane D where the BPS bound is saturated, where the riemannian volume equals the symplectic volume. This is achieved at a point $p \in D$ precisely when the tangent space TD is a complex subspace of TX (this is the Wirtinger inequality). This bound is achieved globaly when D is a holomorphic submanifold of X. (In particular, holomorphic submanifolds minimize volume in their homology class).

We see that a submanifold D defines a BPS particle if it is holomorphic. However, there are more possible boundary conditions. The submanifold can carry a holomorphic line bundle which influences the string, or generally a holomorphic vector bundle. Let me give you the end result of this analysis. To make the BPS states simpler to analyze, we took the limit as the kahler metric ω on X scaled to infinity. This yeilds the following realization of BPS states:

Definition: BPS D-brane for type IIA stirng theory, in the large volume limit: A BPS brane on X is a holomorphic submanifold $D^p \subset X$ with a hermitian vector bundle E whose chern connection has curvature F satisfying:

- $F \in \Omega^{1,1}(D, \mathrm{End}(E))$ (E is a holomorphic vector bundle)
- $\mathrm{Tr}(F) \wedge \omega^{p-1} = \lambda\omega^p$ (Hermitian-yang-mills equation)

Theorem (Donadlson, Uhlembeck, Yau) A holomorphic vector bundle (D^p, E) carries a hermitian yang-mills connection if and only if E is *slope stable*. The slope of E is defined as

$$\mu(E) = \frac{\int_D F \wedge \omega^{p-1}}{\mathrm{rank}(E)} = \frac{\deg E}{\mathrm{rank}\ E}$$

E is slope stable if, for any subbundle F, $\mu(E) > \mu(F)$.

The B-model.

Warning!! Confusingly, the BPS branes in type IIA string theory are B-branes, while the BPS branes in type IIA string theory are A branes.

Now we move these BPS states from the physical, metric dependent type IIA theory to a topological string theory. We do a so-called "B-twist" to produce a 2D topological theory, the B-model, which only depends on the complex structure of X. The boundary conditions form the derived category of coherent sheaves. Let's break this down to its parts:

- Objects: Coherent sheaves. Think holomorphic vector bundles supported on holomorphic submanifolds.
- Morphisms: Say $D = (S, E)$ and $D' = (S', E')$ are two boundary conditions of the above form. The morphisms in the category of boundary conditions $\mathrm{Hom}(D, D')$ counts the number of strips with boundary condition D on one side, and D' on the other. However, in the B-model, supersymmetry restricts the allowed maps Σ we can consider. It forces the maps to be *constant*. So, $\mathrm{Hom}(D, D')$ counts the intersection points of the submanifolds S, S'. Let us give two examples
  - When S, S' intersect transversally, $\mathrm{Hom}(D, D') = |S \cap S'| \cdot \mathrm{rank}(E)\mathrm{rank}(E')$.
  - When $S = S'$, $\mathrm{Hom}(D, D') = H^{0,*}_{\bar\partial}(S, \mathrm{Hom}(E, F)) = \mathrm{Ext}^*(E, F)$
    - This example arises because the supersymmetry operator Q maps to $\bar\partial$. Physical states are Q cohomology.

Konstevich says this is still not big enough. The full set of branes should consist of *complexes* of coherent sheaves, with a differential given by Q. The usual considerations of BRST cohomology tell us to mod out by quasiisomorphism. Hence, the category of boundary conditions the B-model on X is the *Bounded, Derived category of coherent sheaves* $D^\flat(X)$.

The B branes carry a grading from the topological invariatns of the sheaf. These are organized in the Mukai vector. For E a vector bundle on X, define the mukai vector as the chern charecter:

$$\xi(E) = (\mathrm{rank}(E), c_1(E), c_2(E), \dots) \in \bigoplus_k H^{2k}(X)$$

We can extend this to coherent sheaf. Every coherent sheaf $\mathcal{F}$ belongs to a complex

$$0 \to E \to F \to \mathcal{F}$$

for E, F vector bundles. We define the Mukai vector of $\mathcal{F}$ as

$$\xi(\mathcal{F}) = \xi(F) - \xi(E)$$

In physics lingo, these topological invaraiants are the "charges" of the associated particle.

Next, we choose a distinguished subclass of "physical" B-branes. We will be somewhat naieve, and only describe B-branes which are coherent sheaves (i.e complexes of only one term). The "physical" B-branes are the ones which arise as BPS branes from the non-topological type IIA theory. As we saw above, these are B-Branes carrying a Hermitian yang mills connection. This in turn had an algebrogeometric manifestation in terms of slope stability.

Definition: A physical B-brane is a slope-stable coherent sheaf.

Remark:

- If we want to study BPS states outside of the large volume limit (i.e including lower order terms in $1/\mathrm{vol}(X)$), then we deform the Hermitian Yang Mills equation. The soltions of this deformed equation exist whenever our vector bundle is *Geizieker stable*, where we replace slope in the definition of stability with

  $$\lim_{k \to \infty} \chi(E \otimes L^k)$$

  where L is a fixed ample line bundle, and χ is the holomorphic euler charecteristic. This is really the nice notion of stability for sheaves.
- We define the rank of a coherent sheaf to be the coefficent of the highest order term in k, in the expancion of the above holomorphic euler ckarecteristic
- The moduli space of stable sheaves is a natural compactification of the moduli space of stable bundles.

BPS states from BPS branes

I lied to you before. When we consider a particle on $\mathbb{R}^{3,1}$ induced from string theory on $\mathbb{R}^{3,1} \times X$, we can't fix the full D brane in spacetime. Instead, we should think of the D brane as evolving in time with the particle, wiggling around. The initial condition of the D brane only fixes the connected component in the moduli space of branes -- and this is fixed by the mukai vector. (This is why we think of it as "charge").

Now we take the limit scaling up the kahler structure of X, the large volume limit. This means any deviations from the vacuum have a very large energy cost. Essentially, the particle will (classically) never leave the lowst energy state. Here, the minimal enegy states for the cohomology class fixed by the mukai vector are stable coherent sheaves. These are packeged together in a moduli space of stable sheaves $\mathcal{M}(\xi)$ Hence, we get a classical effective field theory for this particle: It is a 1 dimensional field theory, with target $\mathcal{M}(\xi)$.

Now we quantize. We already know the supersymmetric quantum theory of a particle on a manifold. It is given by supersymmetric quantum mechancics! Thus, the full hilbert space for particles with mukai vector ξ is

$$\mathcal{H}_\xi = \Omega^*(\mathcal{M}(\xi))$$

and the subspace of BPS states is

$$\mathcal{H}_\xi^{BPS} = H^*(\mathcal{M}(\xi))$$

On the algebras of BPS states
Proposal (Moore,Harvey): The space of BPS states with mukai vector ξ is the cohomology of the moduli space of stable sheaves with mukai vector ξ

Remark:

- The deformation invariant count of BPS states the $\mathbb{Z}_2$ graded dimension of H^*. In partiuclar the "BPS index" is the *Euler charecteristic* of the moduli space of sheaves $\mathcal{M}(\xi)$. This is called a *Donaldson-Thomas invariant* of X.

BPS states and stability

from nlab, Bridgeland stability conditions
The foucs on stable BPS states can be understood nicely physically. The slope of a coherent sheaf

$$\mu(F) = \frac{\deg(F)}{\operatorname{rank}(F)}$$

should be thought of as a measure of *charge density*. The degree is the charge associated to a D-brane, and the rank is the mass of the D brane (Indeed, we should think of a rank r bundle over a submanifold D as a stack of r copies of D). BPS states are by construction those with minimal mass for their given central charge. That is, their charge to mass ration is as small as possible.

The same is true of slope stable sheaves. This condition means there is no subsheaf $A \hookrightarrow F$ such that $\mu(A) \subset \mu(F)$. In other words, there are no smaller D-branes of higher chagre density. This means that the BPS D-branes are stable. If it were to conserve its charge while splitting into multipile particles, it would increase its mass. Hence, it would not decay at rest. This gives the great coincedence of nominclature: *GIT stability implies physical stability.*

If I have an exact sequence of sheves

$$0 \to A \to B \to C \to 0$$

I should think of this as a expressing B as a bound state of particles A, C. If B is slope stable, then we know

$$\mu(A) < \mu(B) < \mu(C)$$

We should read this as, the charge density of a bound state of A and C is between that of A or C.

BPS algebra

Moore and Harvey argue in On the algebras of BPS states that the BPS states should always form an algebra. They suggest a product

$$R : \mathcal{H}_{BPS} \otimes \mathcal{H}_{BPS} \to \mathcal{H}_{BPS}$$

This is defined by computing the scattering amplitudes of BPS particle. Take a particle of charge Q_1 and a particle of charge Q_2, and throw them at eachother with total energy $Z(Q_1 + Q_2)$, and wait for them to scatter. The resulting state has chage $Q_1 + Q_2$ and mass $M = Z(Q_1 + Q_2)$. Orthogonally projecting this onto $\mathcal{H}_{BPS}$ gives the product structure.

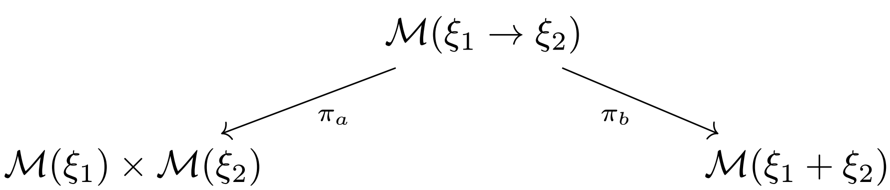
Let's do this more mathematically. For a mukai vector ξ, we define the BPS algerba by $\mathcal{H}_{BPS}(\xi) = H^*(\mathcal{M}(\xi))$. We wish to define a prodcut

$$R : \mathcal{H}_{BPS}(\xi_1) \otimes \mathcal{H}_{BPS}(\xi_2) \to \mathcal{H}_{BPS}(\xi_1 + \xi_2)$$
$$R : H^*(\mathcal{M}(\xi_1)) \otimes H^*(\mathcal{M}(\xi_2)) \to H^*(\mathcal{M}(\xi_1 + \xi_2))$$

We can define this using a hall-type algebra, measuring extentions of sheaves. Define the moduli space of sheaf extentions

$$\mathcal{M}(\xi_1 \to \xi_2) = \{\text{complexes of stable sheaves } 0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \\ \text{such that } \mathrm{ch}(F_1) = \xi_1, \mathrm{ch}(F_2) = \xi_2\}$$

This moduli space comes with two forgetful maps: $\pi_a(F_1 \to F \to F_2) = (F_1, F_2)$ and $\pi_b(F_1 \to F \to F_2) = F$.



Proposal: The BPS algerbra is the push-pull map of cohomology of this diagram

Explicitly, $H^*(\mathcal{M}(\xi))$ is represented by a harmonic form on $\mathcal{M}(\xi)$. For a harmonic form ω_1 on $\mathcal{M}(\xi_1)$ and ω_2 on $\mathcal{M}(\xi_2)$, we define a harmonic form $\pi_a^*(\omega_1, \omega_2)$ on $\mathcal{M}(\xi_1 \to \xi_2)$. then, we take a cohomology pushforward along π_b. That is, we integrate $\pi_a^*(\omega_1, \omega_2)$ along the fibers of π_b, which consist of the moduli of different subsheaves F_1 of $\mathcal{F}$.

This is *Extremly similar to the cohomological hall algebra!* the issue is, it isn't defined. The moduli spaces of sheaves $\mathcal{M}$ can be kind of nasty, and we don't have a good way to take their cohomology. Also, integration along fibers requires a good notion of a virtural fundamental class, which is also technically challenging. This is part of the motivation for Konstevich and Soibelman's work on motivic donaldson thomas invariants. (Its a lot easier to take cohomology if you never forget what your variety is). The trouble boils down to, the moduli space of stable sheaves is an infinite dimensional GIT quoitent (Space of connections modulo gauge transforms). This makes the machinery of stacks hard to apply. Instead, we could try to find a finite dimensional GIT model that gets very close to these moduli spaces. This is the role of Quivers.

Part 3: CoHA is a BPS algebra

(follows section 5.4.1 of D-branes and Mirror symmetry)

Now I want to relate this story to that of Quiver and their moduli of representations.

First, let me describe how this happens physically. From general principles, we expect the type IIA string theory on $\mathbb{R}^{3,1} \times X$, after integrating out X, to give an effective field theory on $\mathbb{R}^{3,1}$. In particular, this should be $d = 4, \mathcal{N} = 1$. We can explicitly describe the possible effective field theories which obey the supersymmetry

Definition: a $\mathcal{N} = 1, d = 4$ effective field theory is a tuple of data (C, H, μ, W) such that

- C is a kahler manifold

- H a compact lie group with isometric, symplectic action on C. The complexification $H_{\mathbb{C}}$ acts holomorphically on C. (The gauge group)
- μ a moment map for the H action
- W a holomorphic, H invariant function (the superpotentail)

Analyzing the action of the supersymmetric theory shows the space of vacuua is the set of points $p \in C$ such that $\mu = 0$ and $\nabla W = 0$. After moding out by gauge transforms, we get that the moduli space of vacuua is

$$\mathcal{M} = \{\nabla W = 0\} //_{\mu=0} H$$

We saw in our arguments already that the cohomology of the moduli space of vacuua gave the space of BPS states.

Example: supersymmetric Quiver gauge theories.
Let Q be a quiver with verticies indexed by a set I, and $\alpha_{i \to j}$ arrows between vertex i and j. Fix a dimension vector $\vec{\gamma}$ and vector spaces V_i, where $\dim(V_i) = \gamma_i$. We define the $\mathcal{N} = 1$ gauge theory:

- $\mathcal{C} = \bigoplus_{i,j \in I} \mathrm{Hom}(V_i, V_j)^{\alpha_{ij}}$ The space of quiver representations
- $H = \bigtimes_i GL(V_i)$, which act on quiver representations by conjugation .
- Fixing the choice of μ fixes the stability condition we use for our GIT quotient $\mathcal{C} // H_{\mathbb{C}}$
- W is a quiver potential. This is a linear combination of the traces along cyclic paths in the quiver. These are the only H invariant holomorphic functions on $\mathcal{C}$

The moduli space of vacuua is called the *Quiver moduli space* $\mathcal{M}(Q, W, \gamma)$. The superpotential imposes relations of the form $\partial_a W = 0$ for all arrows a. If W is defined by a cycle, and a an arrow in the cycle, then $\partial_a W$ is the relation imposed by deleting the arrow a. That is, the composition of all the other arrows in the cycle must be zero. If W does not contain a, then $\partial_a W = 0$. For all other potentials, extend linearly.

EXAMPLE 4.33. In analogy with Example 4.32, consider the quiver with one vertex and three loop arrows a_1, a_2, a_3. Define a superpotential W on this quiver by $W = a_1 a_2 a_3 - a_1 a_3 a_2$. Notice that the permutations (123) and (132) are not cyclic rotations of each other, and hence W is a nonzero element of $A(Q)/[A(Q), A(Q)]$. It is easy to check that the process described above leads to the ideal of relations

$$R_W = \langle\langle a_1 a_2 - a_2 a_1, a_2 a_3 - a_3 a_2, a_3 a_1 - a_1 a_3 \rangle\rangle,$$

and hence the superpotential algebra in this case is just the (commutative) polynomial algebra $\mathbb{C}[a_1, a_2, a_3]$, the ring of functions on affine 3-space. Compare also with Example 5.21.

Superpotential of Donaldson-Thomas theory

Now we construct the effective field theory from string theory in the same framework. For ease of definition, we will stick to a vector bundle on all of X (a D^6 brane). This can presumably be extended to sub-manifolds, and to other coherent sheaves
Example: IIA string theory.
Fix a calabi-yau 3-fold X, a vector bundle E, and a mukai vector $\xi = ch(E)$.

- $\mathcal{C}$ the space of connections on E. The tangent space consists of sections of $\Omega^1(X, \mathrm{ad}(E))$. There is a natural symplectic form, defined by

$$\omega(\alpha, \beta) = \int_X \omega^{d-1} \wedge \mathrm{Tr}(\alpha \wedge \beta)$$

- $H = \mathrm{Aut}(GL(E))$ is gauge transforms of E. The lie algebra h consists of sections of $\Omega^0(X, \mathrm{ad}E)$
- $\mu(\alpha) = F(\alpha) \wedge \omega^{d-1} - \lambda\omega^d$
- $W(A) = \int_X \mathrm{Tr}\left(\bar{\partial} A \wedge A + \frac{2}{3} A \wedge A \wedge A\right) \wedge \Omega$
  We produce the moduli space of hermitian yang mills connections. The moment map for the gauge group action is exactly the hermitian yang mills euqation. Let's explain the role of the superpotential.

$\Omega \in H^0(\Omega^{3,0}(X))$ is a "holomorphic volume form". This is canoncial on X calabi yau, because $K_X = \Omega^{3,0}(X) = \mathcal{O}_X$ has a global generating section. The middle part of the equation is very similar to the chern-simons functional on a 3 manifold M:

$$CS(A) = \int_M \mathrm{Tr}(\mathrm{d}A \wedge A + \frac{2}{3} A \wedge A \wedge A)$$

The only difference is trading the exterior derivative with $\bar{\partial}$. In this sense, $W(A)$ is a *Holomorphic chern simons functional*. The space of Vacuua consists of the critical points of W. The chern simons functional has derivative

$$\mathrm{d}CS(a)|_{A_0} = \int_M \mathrm{Tr}(a \wedge (d + A_0)^2) = \int_M \mathrm{Tr}(a \wedge F_{A_0})$$

hence the critical points are flat connections $F_{A_0} = 0$. Analagously, the derivative of the holomorphic Chern-Simons functional is

$$\mathrm{d}W(a)|_{A_0} = \int_X \mathrm{Tr}(a \wedge (\bar{\partial} + A_0)^2) \wedge \Omega = \int_X \mathrm{Tr}(a \wedge F^{0,2}_{A_0}) \wedge \Omega$$

This vanishes when $F^{0,2}$ vanishes. Hence, *the critical points of the holomorphic chern simons functional are holomorphic connections*

All together, the vacuum moduli space of this theory is the moduli space of holomorphic connections with a hermitian-yang-mills connection, modulo gauge transform. This is isomorphic to the moduli space of stable holomorphic vector bundles.

To analyze the holomorphic chern simons perturbativly, choose a vacuum (some critical point of the chern simons functional) and consider the expansion around that point. The defomration space is $H^1(X, \mathrm{End}(E))$. (???). Then, the holomorphic chern simons superpotential yeilds a cubic term. There are higher order terms, encoding the A_∞ structure of $D(X)$. See the discussion in section 3 of Superpotentials for Quiver Gauge Theories by Paul S. Aspinwall, Lukasz M. Fidkowski.

Remark:

- The euler charecteristic of this vacuum moduli space is called the "donaldson thomas invariant". It was origionally introduced in the framework of this "holomorphic casson invariant".
- In thomas's origional paper, he proceeds to compactly this moduli space (using Geizieker stable sheaves) and constructs a virtural class, enough to compute the euler charecteristic. See A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on K3 fibrations.

From branes to quivers

For general physics reasons, the effective theory of the D branes should be a quiver gauge theory. This is described in 5.4.2.1 of D-branes and mirror symmetry, but i just don't understand.

Example:

A quiver is a combinatorial model (with generators / relations) of a dg category (??). They come up in categories of coherent sheaves when the sheaves are particularly combinatorial.

Definition: a *Full, strong exceptional collection* $E_1, \ldots, E_n$ is a collection of objects which generate $D(X)$ such that

- $Hom(E_i, E_i) = \mathbb{C}$
- $Hom(E_i, E_j) = 0$ for $i < j$

- $\mathrm{Ext}^k(E_i, E_j) = 0$ for $k \neq 0$

$\mathrm{Ext}^k(E_i, E_j) = 0$ for $i \neq j$, and $\mathrm{Ext}^k(E_i, E_i) = \mathbb{C}$ if $k = 0$, zero otherwise.

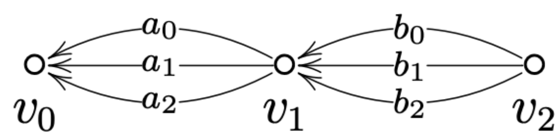
From this data, we construct a quiver Q. The verticies come from the set E_i. The number of paths from i to j is the dimension of $\mathrm{Hom}(E_i, E_j)$.

Theorem (Bondal): $D(X) \cong D(A - mod)$

In other words, using a full exceptional collection we can repoduce the full derived category from a quiver.
See [Bondal: Helices, representations of quivers and Koszul algebras]

Example: $\mathbb{P}^2$ has a full exceptional collection $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$.
Proof:
idk a lot of computation.

The associated quiver has 3 notes, $E_0 = \mathcal{O}, E_1 = \mathcal{O}(1), E_2 = \mathcal{O}(2)$, with 3 arrows each going from one to the next. There is a relation too, which can be expressed as the critical set of a superpotential.



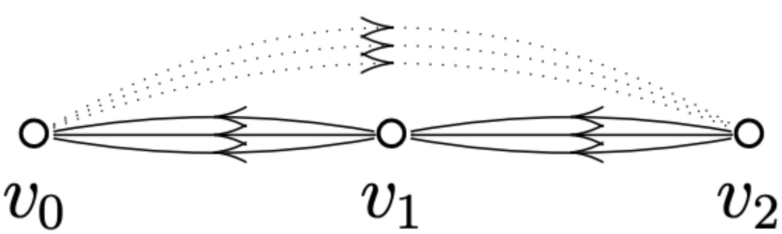
with relations $a_\alpha b_\beta - a_\beta b_\alpha$.

Usually we can only find exceptional collections for fano varieties. But, we build bundles over these fano varieties to make them Calabi-yau. Then, the understanding of the derived category of sheaves will transfer over to the calabi-yau.

In this case, we consider the total space of the anticanonical bundle $\omega = \mathcal{O}_{\mathbb{P}^2}(-3)$, with projection $\pi : \omega \to \mathbb{P}^2$. By construction, this space has trivial canonical bundle

$$K_\omega = \pi^*(K_{\mathbb{P}^2} \otimes K^*_{\mathbb{P}^2}) = \mathcal{O}_\omega$$

We consider the sheaves $\pi^*(E_i)$. The pulled back sheaves are no longer exceptional. Indeed, the freedom of $\mathbb{P}^2$ sitting inside its canonical bundle allows Ext^1 groups to appear. In particular, $\mathrm{Ext}^1(\pi^*(\mathcal{O}(2), \pi^*(\mathcal{O})) \neq 0$. Using this, we "complete" the quiver



Bridgland constructed a algebra B built out of the uncompleted quiver, such that $D(B - \mathrm{mod}) \cong D(\omega)$. He did this in T-STRUCTURES ON SOME LOCAL CALABI-YAU VARIETIES. Later, in Superpotentials for Quiver Gauge Theories , Aspinwall and Fidkowski proved that the B is the path algebra of the completed quiver (Above), with relations given by a superpotential. The superpotential is given by "closing up" the relations from the original quiver from the eceptional collection. To prove this, they used the A_∞ structure and the relation to the full superopotential on the moduli space of sheaves.

See discussion in Moduli Spaces for D-branes at the Tip of a Cone by Aaron Bergman and Nicholas Proudfoot.

Papers

Charge of a D-brane in B model is the generalized mukai vector of a sheaf:
Green, Harvey, Moore, I-Brane Inflow and Anomalous Couplings on D-Branes

Moduli space of stable bundles is compactified by Geizieker-semistable sheaves :
GIESEKER, On the moduli space of vector bundles on an algebraic surface.
Geiseker semistability: Fix an ample line bundle L. The "slope" is

$$\mu(E)(k) = \chi(E \otimes L^k)/\mathrm{rank}(E)$$

where χ is the holomorphic euler charecteristic, or the Riemann Roch number.

Moduli of sheaves

- **Authors:** Mestrano, Simpson
- **Year:** 2016
- **Summary:** Very nice survey on what we know about moduli spaces of sheaves. Defines stability in section 2. Breifly discusses wall crossing and Konstevich-Soibelman in section 13.

On the algebras of BPS states

- **Authors:** Harvey Moore
- **Year:** 1997
- **Summary:** Suggests a defintition of the aalgebra of BPS states. In type IIA string theory, the BPS states are harmonic forms on the moduli space of stable sheaves of a Calabi-yau 3-fold. They suggest the BPS product is given by the push-pull formula on the cohomology of sheaves, a hall algebra type structure. They call this the "correspondence conjecture"

Motivic degree zero Donaldson-Thomas invariants

- **Authors:** Kai Behrend, Jim Bryan, Balazs Szendroi
- **Year:** 2009
- **Summary:** Gives a good summary explintion of motivic donaldson thomas invariants, and the motovic milnor fiber.

Donaldson-Thomas Type Invariants via Microlocal Geometry

- **Authors:** K. Behrend
- **Year:** 2005
- **Summary:** First exposes the motivic notion of donaldson thomas invariants. Realizes the donaldson thomas moduli space as the critical locus of the holomorphic chern simons function (a classical construction), then uses essentially the motivic milnor fiber.

AN INTRODUCTION TO (MOTIVIC) DONALDSON-THOMAS THEORY

- **Authors:** SVEN MEINHARDT
- **Year:** 2017
- **Summary:** Expository introduction to motivic donaldson thomas invariants. Does everything in the example of quivers with potential, as a model of the more general theory.

Refined, Motivic, and Quantum

- **Authors:** Tudor Dimofte and Sergei Gukov
- **Year:** 2009

- **Summary:** Relats Konstevich and Soibelmans motiovic wall crossing formula with the refined count of BPS states, and their wall crossing.

Superpotentials for Quiver Gauge Theories

- **Authors:** Paul S. Aspinwall and Lukasz M. Fidkowski
- **Year:** 2005
- **Summary:** Talks about quivers and how to make them and their superpotentials from brane setups. Describes some equvielnces of the categroy of modules of the path algebra with derived categories of coherent sheaves,