A Chart for Getting into 3CY Paper.

To a SUSY QFT, there is an algebra of "BPS" states.

[KS 11] Models this algebra by constructing "COHA"

Roughly speaking,

Repn. of quivers $\to$ CoHA.

"smooth algebras" $\to$ CoHA.

$\cup$

Finitely presented algebras $\to$ CoHA.

$\cdots$ $\to$ CoHA.

Key Observation

[KS08] For each CoHA, its "motivic DT-series" $\rightsquigarrow$ ind-constructible 3CY cat.

2 versions of CoHA $\begin{cases} \text{Rapid decay} \\ \text{Critical} \end{cases}$ Related in a non-trivial way.

Conj. All (?) 3CY cat $\mathcal{C}$ arise from some CoHA $\mathcal{H}_{\mathcal{C}}$.

Pf: ① Define $\mathcal{H}_{\mathcal{C}}$ [...]

② $\mathcal{H}_{\mathcal{C}} \otimes \mathcal{H}_{\mathcal{C}} \to \mathcal{H}_{\mathcal{C}}$ [KPS 24] Thm A

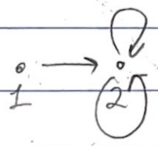
Today: Practical (i.e. "Rapid Decay" version) intro. to CoHA.

§1. CoHA of a Quiver
§2 CoHA of a Quiver w/ potential

Disclaimer: [KS11] is extremely dense. Below is what I'm able to extract in ~ 12 hours as a complete beginner in CoHA.

§1

Let Q be a quiver, I set of vertices, $a_{ij} \geq 0$ the # of arrows from $i \to j$



$a_{11} = 0$ $\quad$ $a_{21} = 0$ $\quad$ $I = \{1,2\}$

$a_{12} = 1$ $\quad$ $a_{22} = 2$.

A "representation" of Q is a set of vector spaces E_i, and maps $E_i \to E_j$ for each arrow.

A "dimension vector" γ of a repn. M of Q is a tuple $\gamma = (\dim E_i)$, $i \in I$.

The space of all repn. of Q (dim $= \gamma$) in $\mathbb{C}$-vector spaces is

$$M_\gamma = \prod_{i,j,a_{ij}} \mathrm{Map}(\mathbb{C}^{\gamma^i}, \mathbb{C}^{\gamma^j})$$

$$\cong \prod_{i,j\, a_{ij}} \mathbb{C}^{\gamma^i \gamma^j} \quad \cong \prod_{i,j} \mathbb{C}^{a_{ij}\gamma^i\gamma^j}$$

Obviously, the group
$$G_\gamma := \prod_{i\in I} GL(\gamma^i; \mathbb{C})$$
acts on M_γ by conjugation of each map (aka change of basis).

Thus the quotient stack M_γ / G_γ is the stack of repn of Q of dim. γ.

Note that M_γ is a complex algebraic, $G_\gamma \curvearrowright M_\gamma$.
One considers the (Betti) equivariant cohomology of M_γ,
$$\mathcal{H}_\gamma := H^\circ_{G_\gamma}(M_\gamma) = H^\circ(EG_\gamma \times_{G_\gamma} M_\gamma)$$
(Coeff. here are all $\mathbb{Z}$)

Sum over the dimension vector γ results in a bi-graded abelian group
$$\mathcal{H} := \bigoplus_\gamma \mathcal{H}_\gamma$$
where the gradings are by γ and the usual cohomological degree.

Claim: $\mathcal{H}$ has the structure of an associative, unital algebra. i.e. $\exists$ a map $m: \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$.

Def. $\mathcal{H}$ is the CoHA of the quiver Q.

- Abstractly, the multiplication is given by the following map

$$m_{\gamma_1,\gamma_2} : \mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2} \longrightarrow \mathcal{H}_{\gamma_1+\gamma_2}$$

behaves as expected for the γ-grading.

(But not for the cohomological grading! later)

which is a composition of 4 maps.

Let γ_1, γ_2 be dim. vectors.

M_{γ_1,γ_2} the space of repn. of Q in $\mathbb{C}$ coordinate spaces of dim $\gamma := \gamma_1 + \gamma_2$, such that the standard subspaces of dim γ_i form a sub-rep.

e.g.
$$M_{\gamma_1+\gamma_2} = \left\{ \left(\begin{array}{c} \\ \left[\ * \ \right]^{\gamma_1^2+\gamma_2^2} \\ \gamma_1^1+\gamma_2^1 \end{array} , \begin{array}{c} \\ \left[\ * \ \right]^{\gamma_1^2+\gamma_1^2} \\ \gamma_1^2+\gamma_1^2 \end{array} \right) \right\}.$$

for $Q = \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \circlearrowleft$

$$M_{\gamma_1,\gamma_2} = \left\{ \left(\begin{array}{cc|c} \begin{bmatrix} \square & * \\ 0 & \square \end{bmatrix} & \begin{array}{c} \gamma_1^2 \\ \gamma_2^2 \end{array} \\ \gamma_1^1 \quad \gamma_2^1 & \end{array} , \begin{array}{cc} \begin{array}{c} \gamma_1^2 \\ \gamma_2^2 \end{array} & \begin{bmatrix} \square & * \\ 0 & \square \end{bmatrix} \\ & \gamma_1^2 \quad \gamma_2^2 \end{array} \right) \right\}.$$

Let $G_{\gamma_1,\gamma_2} \subset G_\gamma$ be the subgroup preserving the subspaces $\mathbb{C}^{\gamma_i}$.

i.e. $G_{\gamma_1,\gamma_2} = \left\{ \begin{array}{c} \begin{bmatrix} \square & * \\ 0 & \square \end{bmatrix}^{\gamma_1^i} \\ \gamma_1^i \end{array} \right\} \subset \prod_{i \in I} GL(\gamma^i; \mathbb{C})$

The 4 maps giving the multip. are

① $\mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2} = H^{\bullet}_{G_{\gamma_1}}(M_{\gamma_1}) \otimes H^{\bullet}_{G_{\gamma_2}}(M_{\gamma_2}) \longrightarrow H^{\bullet}_{G_{\gamma_1} \times G_{\gamma_2}}(M_{\gamma_1} \times M_{\gamma_2})$

equivariant version of Künneth morphism

② $H^{\bullet}_{G_{\gamma_1} \times G_{\gamma_2}}(M_{\gamma_1} \times M_{\gamma_2}) \xrightarrow{\sim} H^{\bullet}_{G_{\gamma_1,\gamma_2}}(M_{\gamma_1,\gamma_2})$

induced by homotopy equiv. of the projections

$$M_{\gamma_1} \times M_{\gamma_2} \overset{\sim}{\twoheadleftarrow} M_{\gamma_1,\gamma_2}$$

$$G_{\gamma_1} \times G_{\gamma_2} \overset{\sim}{\twoheadleftarrow} G_{\gamma_1,\gamma_2}$$

both given by sketches

$$\left(\boxed{1}, \boxed{2} \right) \longleftarrow \begin{pmatrix} \boxed{1} & * \\ 0 & \boxed{2} \end{pmatrix}$$

homotopy to the zero block

③ $H^{\bullet}_{G_{\gamma_1,\gamma_2}}(M_{\gamma_1,\gamma_2}) \longrightarrow H^{\bullet + 2C_1}_{G_{\gamma_1,\gamma_2}}(M_{\gamma})$

is the pushforward along $M_{\gamma_1,\gamma_2} \hookrightarrow M_{\gamma}$

$$\begin{pmatrix} \square & * \\ & \square \end{pmatrix} \longmapsto \begin{pmatrix} \square & * \\ & \square \end{pmatrix}$$

both M_{γ_1,γ_2} and M_γ are G_{γ_1,γ_2}-sets. So get the equivariant pushforward.

the degree shift is given by

$$c_1 := \dim_{\mathbb{C}} M_\gamma - \dim_{\mathbb{C}} M_{\gamma_1,\gamma_2} = \sum_{i,j\in I} a_{ij}\gamma_1^j \gamma_2^i$$

count the size of the zero blocks.

? (4) $H^{\bullet+2c_1}_{G_{\gamma_1,\gamma_2}}(M_\gamma) \longrightarrow H^{\bullet+2c_1+2c_2}_{G_\gamma}(M_\gamma)$

Pushforward associated w/ "the fund. class of compact complex mfd $G_\gamma/G_{\gamma_1,\gamma_2} = \prod_{i\in I} Gr(\gamma_1^i, \mathbb{C}^{\gamma^i})$"

$$c_2 := -\dim_{\mathbb{C}}(G_\gamma/G_{\gamma_1,\gamma_2}) = -\sum_{i\in I}\gamma_1^i\gamma_2^i.$$

These 4 maps compose to

$$\begin{tikzcd}
H^\bullet_{G_{\gamma_1}}(M_{\gamma_1}) \otimes H^\bullet_{G_{\gamma_2}}(M_{\gamma_2}) \arrow[r, "①"] \arrow[dr, dashed, "m_{\gamma_1,\gamma_2}"'] & H^\bullet_{G_{\gamma_1}\times G_{\gamma_2}}(M_{\gamma_1}\times M_{\gamma_2}) \arrow[r, "\cong", "②"'] & H^\bullet_{G_{\gamma_1,\gamma_2}}(M_{\gamma_1,\gamma_2}) \arrow[d, "③"] \\
& H^{\bullet+2c_1+2c_2}_{G_\gamma}(M_\gamma) & H^{\bullet+2c_1}_{G_{\gamma_1,\gamma_2}}(M_\gamma) \arrow[l, "④"']
\end{tikzcd}$$

$$m: \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}, \quad m := \sum_{\gamma_1,\gamma_2} m_{\gamma_1,\gamma_2}.$$

Properties: m is associative, unital,

Rmk: $2(G+Q) = -2\chi_Q(\gamma_1,\gamma_2)$
where $\chi_Q(\gamma_1,\gamma_2)$ is the "Euler form" on $K_0(\mathrm{Rep}^{f.\dim} Q)$.

$$\chi_Q(\gamma_1,\gamma_2) = \dim \mathrm{Hom}(E_1,E_2) - \dim \mathrm{Ext}^1(E_1,E_2)$$
$$= \dim \mathrm{Ext}^{\bullet}(E_1,E_2)$$
for any $E_{1,2}$ of Q of dim γ_1,γ_2.

- Alternative formulations

There is a natural morphism of smooth Artin stacks
$$M_{\gamma_1,\gamma_2}/G_{\gamma_1,\gamma_2} \longrightarrow M_\gamma/G_\gamma$$
this map turns out to be proper, hence inducing pushforward on cohomology.

There is also a homotopy equivalence
$$M_{\gamma_1}/G_{\gamma_1} \times M_{\gamma_2}/G_{\gamma_2} \xleftarrow{\sim} M_{\gamma_1,\gamma_2}/G_{\gamma_1,\gamma_2}$$
So can pull back cohomology along this

$$\begin{tikzcd} & M_{\gamma_1,\gamma_2}/G_{\gamma_1,\gamma_2} \arrow[dl] \arrow[dr] & \\ M_{\gamma_1}/G_{\gamma_1} \times M_{\gamma_2}/G_{\gamma_2} & & M_\gamma/G_\gamma \end{tikzcd}$$
gives m_{γ_1,γ_2}.

- Formula

M_γ is contractible, so

$$\mathcal{H}_\gamma = H^\bullet(EG_\gamma \times_{G_\gamma} M_\gamma) = H^\bullet(BG_\gamma) = H^\bullet\Big(B\big(\prod_{i\in I} GL(\gamma^i,\mathbb{C})\big)\Big).$$

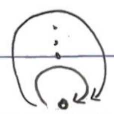
But $H^\bullet(BGL_n(\mathbb{C})) \cong \mathrm{Sym}\,\mathbb{Z}[x_1,\dots,x_n]$, $\deg x_i = 2$.

Thus $\mathcal{H}_\gamma = \mathbb{Z}[x_{i,\alpha}] \Big/ \prod_{i\in I} \mathrm{Sym}_{\gamma^i}$, $i\in I$, $\alpha = 1,\dots,\gamma^i$

— awkward notations —

Thm Let $f_1 \in \mathcal{H}_{\gamma_1}$, $f_2 \in \mathcal{H}_{\gamma_2}$, $g = f_1 \cdot f_2 \in \mathcal{H}_{\gamma_1+\gamma_2}$. Let $\gamma = \gamma_1 + \gamma_2$. Then g is given by

$$g\left((x_{i,\alpha})_{i\in I}^{\alpha\in\{1,\dots,\gamma^i\}}\right) = \sum_{\substack{\text{shuffles} \\ \text{of } x'_{i,\alpha},\, x''_{i\alpha}}} f_1(x'_{i\alpha}) f_2(x''_{i\alpha}) \frac{\prod_{i,j,\alpha_1,\alpha_2} (x''_{j\alpha_2} - x'_{i\alpha_1})^{a_{ij}}}{\prod_{i,\alpha_1,\alpha_2} (x''_{i,\alpha_2} - x'_{i\alpha_1})}$$

e.g. Q_d  d-many loops. $d \geq 0$. $\gamma_1^1 = n$, $\gamma_2^1 = m$.
$\gamma^1 = n+m$

$$(f_1 \cdot f_2)(x_1, x_2, \dots, x_n, \dots, x_{n+m})$$
$$= \sum_{\substack{i_1<\dots<i_n \\ j_1<\dots<j_m \\ \{i_1,\dots,i_n,j_1,\dots,j_m\} \\ =\{1,\dots,n+m\}}} f_1(x_{i_1},\dots,x_{i_n}) f_2(x_{j_1},\dots,x_{j_m}) \prod_{l=1}^{m}\prod_{k=1}^{n} (x_{j_l} - x_{i_k})^{d-1}$$

Say for e.g. $\gamma_1' = \gamma_2' = 1$. $\gamma' = 2$.

$$\begin{array}{ccccc} \mathcal{H}_1 & \otimes & \mathcal{H}_1 & \to & \mathcal{H}_2 \cong \mathrm{Sym}\,\mathbb{Z}[x,y] \\ \wr\wr & & \wr\wr & & \\ \mathrm{Sym}\,\mathbb{Z}[x] & & \mathrm{Sym}\,\mathbb{Z}[y] & & \end{array}$$

$$(f_1 \cdot f_2)(x,y) = f_1(x) f_2(y) (y-x)^{d-1} + f_1(y) f_2(x) (x-y)^{d-1} \in \mathrm{Sym}\,\mathbb{Z}[x,y].$$

– claim: d odd : $\mathcal{H}$ is comm.
even : supercomm.

Supercomm
how?

Q. $d = 0$,

$$(f_2 \cdot f_1)(x,y) = f_2(x) f_1(y) (y-x)^{-1} + f_2(y) f_1(x) (x-y)^{-1}.$$

$f_1 \cdot f_2 = (-1)^{|?||?|} f_2 \cdot f_1$.

Homogeneous deg k n-var – $(n \cdot 2k + (1-d)n^2)$

$d=0$: $\mathcal{H}$ gen. by odd elements of deg $(1,1)$, $(1,3)$, $(1,5)$...
$k = \quad 0, \quad 1, \quad 2, \ldots$

(Time Permitting)

§2 Defn Smooth Algebras

R assoc. unital $/k$ is "smooth" if R is fin. gen. and the (R,R)-bimod

$$\Omega^1_R := \ker(R \otimes_k R \xrightarrow{m} R)$$

is projective.

e.g. Matrix algebras, algebras of functions on smooth aff curves, Path algebras of finite quivers, etc.

Rmk Morita inv.

Defn: Let I be a finite set.

R assoc. unital $/k$ is I-bigraded if

$$R = \bigoplus_{i,j \in I} R_{ij}, \quad R_{ij} \cdot R_{jk} \subset R_{ik}.$$

If R smooth, then any fin. dim repn E of R decomposes into a direct sum

(?) $$E = \bigoplus_{i \in I} E_i.$$

Let $\gamma = (\gamma^i)_{i \in I} \in \mathbb{Z}^I_{\geq 0}$ be a dim vector.

M_γ the scheme of Rep of R in coord spaces $E_i = k^{\gamma^i}$, $i \in I$ is smooth aff.

Observation:

Any choice of a finite set of I-bigraded generators of R gives a closed embedding

$$M_\gamma \hookrightarrow M_\gamma^Q$$

for some Q quiver, I vertices.

$\boxed{\text{Assumption}}$ $k=\bar{k}$, $\exists$ bilinear form $\chi_R : \mathbb{Z}^I \otimes \mathbb{Z}^I \to \mathbb{Z}$ s.t. any two γ_1, γ_2 dim. vectors and any two repn $E_1 \in M_{\gamma_1}$, $E_2 \in M_{\gamma_2}$, we have

$$\dim \operatorname{Hom}(E_1, E_2) - \dim \operatorname{Ext}^1(E_1, E_2) = \chi_R(\gamma_1, \gamma_2)$$

$$\underset{\text{implies}}{\overset{\text{Apparently}}{\Longrightarrow}} \dim M_\gamma = -\chi_R(\gamma, \gamma) + \sum_i (\gamma^i)^2.$$

If $R = kQ_{(p_1 p_2, \dots, p_n)}$ (finite localization), then $\chi_R = \chi_Q$,

$$M_\gamma \underset{\substack{\text{open} \\ \text{Zariski}}}{\subset} M_\gamma^Q = \mathbb{A}_k^{\sum_{ij} a_{ij} \gamma^i \gamma^j}.$$

Can construct analogous CoHA.

Defn Rapid Decay Cohomology

X complex algebraic variety, $f \in \mathcal{O}(X)$ regular map $f : X \to \mathbb{C}$,

$$H^\bullet(X, f) := \lim_{t \to \infty} H^\bullet(X, f^{-1}(\ \ ))$$

(I tried to make sense of this, but failed.)

Properties:

- $\pi : Y \to X$, $f_X : X \to \mathbb{C}$, $f_Y : Y \to \mathbb{C}$, $f_Y = \pi^* f_X$,

then there is pull back

$$\pi^* : H^\bullet(X, f_X) \to H^\bullet(Y, f_Y)$$

- π proper, X, Y smooth, then can push forward

$$\pi_* : H^\bullet(Y, f_Y) \to H^{\bullet + 2(\dim_{\mathbb{C}} X - \dim_{\mathbb{C}} Y)}(X, f_X)$$

- Künneth morphism

$$\otimes : H^\bullet(X, f_X) \otimes H^\bullet(Y, f_Y) \to H^\bullet(X \times Y, f_X \boxplus f_Y)$$

where $\boxplus$ is "Thom-Sebastiani sum"

$$f_X \boxplus f_Y := pr^*_{X \times Y \to X} f_X + pr^*_{X \times Y \to Y} f_Y.$$

Let $W \in R/[R,R]$, $W_\gamma : M_\gamma \to \mathbb{C}$ is G_γ-equiv.

$$E \mapsto tr_E(W)$$

follow the case of a quiver to define

$$\mathcal{H}^W = \bigoplus_\gamma \mathcal{H}^W_\gamma, \quad \mathcal{H}^W_\gamma := H^\bullet_{G_\gamma}(M_\gamma, W_\gamma) = H^\bullet(M_\gamma / G_\gamma, W_\gamma)$$

If $R = kQ$, $W \in kQ/[kQ, kQ]$ is a formal sum of cyclic paths in Q.

Properties Same as before.

e.g. $Q_1 =$  $\quad W = \sum_{i=0}^{N} c_i l^i$, $c_N \neq 0$ deg N poly.

γ is just an integer.

To check: $N=0$, $\mathcal{H}^W$ = CoHA of quiver Q.

$N=1$, $\mathcal{H}^W = \mathcal{H}_0 = \mathbb{Z}$.

$N=2$, $\mathcal{H}^W_n \simeq H^\cdot(BGL(n,\mathbb{C}))[-n^2]$

$\mathcal{H} \simeq$ exterior algebra w/ infinitely many generators.

this guy

$N \geq 3$, $\mathcal{H}^W \simeq \bigotimes^{N-1} \mathcal{H}^W_n$.

Hopefully the next speaker would introduce the "critical CoHA" and how CoHA gives rise to 3 CY. or motivic DT-series.