A warp-speed overview of DAG

DAG: take suitable "homotopical" replacements of everything in classical AG

Comm. ring $\rightsquigarrow$ "derived comm. ring" (cdga / simplicial comm. ring / $\mathbb{E}_\infty$ ring spectrum)

Ab. cat $QCoh(X) \rightsquigarrow$ derived dg-/∞-cat $D_{qc}(X)$

Stacks $Aff_k^{op} \to Grpd \rightsquigarrow$ derived stacks $dAff_k^{op} \to Spaces$

Cotangent bundle $\Omega^1_{X/Y} \rightsquigarrow$ cotangent complex $\mathbb{L}_{X/Y}$

This requires "working htpically" but has many benefits

Ex: can remove Tor-independence /... hypotheses on base change thms

Ex: "Correct" intersections (explaining Serre intersection fla)

This is especially important in physical mathematics

$Crit(W) = \Gamma_{dW} \cap \{0\text{-section}\} \subset T^*X$ must be interpreted in derived sense in general

Ex: If X is a der. stk $/k$ locally of finite pres (in htpical sense, i.e. compact), then $\mathbb{L}_{X/\mathrm{Spec}\,k}$ is perfect! (Behaves like VB, e.g. is dualizable in $D_{qc}(X)$)

"Hidden smoothness": Many moduli stacks behave like smooth stacks when considered w/ their full derived structure. Can get VFCs, ...

How to work with DAG: Fake it!

A few key principles to remember:

- Always work homotopically. "$=$" should usually be replaced by "$\simeq$".
- This often results in "properties" becoming "structure" (e.g. commutativity $\rightsquigarrow \mathbb{E}_\infty$)
- "Derived" info lies to the left (negative cohom. degs), "stacky" info to the right (positive)
- Prove results for derived Artin stacks by "bootstrapping" from cdgas & using descent

Shifted Symplectic Forms - What?

Recall: An (algebraic) symplectic form on a smooth algebraic variety X is a closed, nondegenerate 2-form ω on X.

(Note: We're interested in algebraic/holomorphic geometry. This notion of "symplectic" is closer to "hyperkähler" than it is to smooth manifold theorists' "symplectic".)

Spelling things out: we want $\omega \in \Gamma(X, \Lambda^2 \Omega_X)$ such that $d\omega = 0$ (closed) and $v \mapsto \omega(v, -)$ induces $\Theta_X \xrightarrow{\sim} \Omega_X$ (where Θ is tangent sheaf). (nondegen)

Generalizing "nondegenerate 2-forms" to derived Artin stacks is "easy" - ask for $\omega \in \Gamma(X, \Lambda^2 \mathbb{L}_X)$ such that ω induces $\mathbb{T}_X \xrightarrow{\sim} \mathbb{L}_X$.

Note: $\mathbb{L}_X$ is a <u>complex</u>, so we can instead ask for "n.d. 2-forms" of any degree n. These are $\omega \in \Gamma(X, \Lambda^2 \mathbb{L}_X[n])$ inducing $\mathbb{T}_X \xrightarrow{\sim} \mathbb{L}_X[n]$.

Ex: Let G be a reductive group, and let $\pi: pt \to BG$ be the universal quotient map. The conormal fiber sequence gives $\pi^* \mathbb{L}_{BG} \simeq \mathrm{fib}(\mathbb{L}_{pt} \to \mathbb{L}_{pt/BG}) \simeq \mathrm{fib}(0 \to \mathfrak{g}^\vee) \simeq \mathfrak{g}^\vee[-1]$

Taking $n=2$, we see $\Gamma(BG, \Lambda^2 \mathbb{L}_{BG}[2]) \simeq \Gamma(BG, \pi^* \Lambda^2 \mathbb{L}_{BG}[2])^G \simeq (\mathrm{Sym}^2 \mathfrak{g}^\vee)^G$

That is, (n.d.) 2-forms of degree 2 on BG are the same as (n.d.) invariant <u>symmetric</u> bilinear forms on $\mathfrak{g}$! Killing form gives an example if $\mathfrak{g}$ semisimple.

"Closedness" is harder to define. First need a notion of "de Rham complex". Then need to replace "$d\omega = 0$" by "$d\omega \simeq 0$", turning closedness from <u>property</u> into <u>structure</u> (data of path witnessing $\simeq$).

We'll postpone this to a later week, but we can at least state:

Def: A n-shifted symplectic form on a derived Artin stack X is a closed 2-form ω of degree n on X such that the underlying 2-form (of degree n) is nondegenerate.

Ex: Choice of n.d. invariant bilinear form on $\mathfrak{g}$ gives 2-shifted symplectic form on BG.

Shifted Symplectic Forms - Why? (ignoring size issues for brevity!)

One viewpt: many interesting geometric constructions on symplectic objects leave the 0-shifted world but remain in the broader shifted symplectic world

Ex: If X and Y are "Lagrangians" in n-symp. Z, then $X \times_Z Y$ is $(n-1)$-symp

In particular, Lagrangian intersections in a 0-symp. stack are (-1)-symp.

Ex: If X is "d-oriented" (e.g. d-dim'l Calabi-Yau variety, or Betti stack of d-dim'l oriented manifold), Y is n-symp, and the mapping stack $\mathrm{Map}(X,Y)$ is "small enough," then $\mathrm{Map}(X,Y)$ is $(n-d)$-symp.

In particular, $\mathrm{Map}(X, BG)$ is $(2-d)$-symp. For $d=2$, we get 0-symp struct!

Another viewpt: Flexibility afforded by shifts relates to that afforded by locality in TFT.

Safronov: Geometric quantization of n-symp stack (+extra data) is n-category

Calaque-Haugseng-Scheimbauer: AKSZ construction of "semi-classical" TFT $\mathrm{Map}(-,X): \mathrm{Bord}_{0,d}^{or} \to \mathrm{Lag}_d^n$ for X n-symp.

($\Rightarrow$ semiclassical versions of Chern-Simons, Rozansky-Witten theories)

One last viewpt: (-1)-symp. stacks control interesting invariants

PTVV: (-1)-symp structs $\Rightarrow$ sym. obstruction theories

These lead to virtual fundamental classes, which in turn give Gromov-Witten invariants

Several authors: Construction of Donaldson-Thomas sheaves

These categorify and explain phenomena surrounding cohomological DT invariants

Goal of this seminar: understand these & related "cohomological Hall algs"