① Some things about cotangent complexes it might be good to know

Recall: A derived comm. ring, $M \in \mathrm{Mod}_A^{\leq 0}$ $\Rightarrow$ get algebra structure on $A \oplus M$ via usual mult on A, $A \curvearrowright M$, and $m^2 = 0$ for $m \in M$

For $\mathcal{X} : \underline{d\mathrm{Comm}_k} \to \underline{\mathcal{S}}$ a derived stack, the (absolute) cotangent complex of $\mathcal{X}$ is $\mathbb{L}_{\mathcal{X}} \in D_{qc}(\mathcal{X})$ satisfying $(\forall A, M)$

$$\mathcal{X}(A \oplus M) \times_{\mathcal{X}(A), x} pt \simeq \Omega^\infty \mathrm{Hom}_A(x^* \mathbb{L}_{\mathcal{X}}, M)$$

Ω^∞ is just turning RHS from cx into space

Space of lifts
$$\begin{tikzcd} \mathrm{Spec}\, A \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{Spec}\, A \oplus M \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}\, k \end{tikzcd}$$

Compare this def'n w/ formal smoothness
Formal smoothness: existence of lifts for <u>arbitrary</u> square-0 extensions
This: finer info, but only for <u>split</u> square-0 exts

<u>Prop</u>: For $\mathcal{X}$ derived Artin stack, $\mathbb{L}_{\mathcal{X}}$ exists & is unique

Can also relativize: for $f : \mathcal{X} \to \mathcal{Y}$, have $f^* \mathbb{L}_{\mathcal{Y}} \to \mathbb{L}_{\mathcal{X}} \to \mathbb{L}_{\mathcal{X}/\mathcal{Y}} \to$

More generally:
$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr] & & \mathcal{Y} \arrow[dl] \\ & \mathcal{Z} & \end{tikzcd}$$
gives $f^* \mathbb{L}_{\mathcal{Y}/\mathcal{Z}} \to \mathbb{L}_{\mathcal{X}/\mathcal{Y}} \to \mathbb{L}_{\mathcal{X}/\mathcal{Z}} \to$
(follows from above)

Prop: For Cartesian
$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X} \arrow[d, "p"] \\ \mathcal{Y}' \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$
have $\mathbb{L}_{\mathcal{X}'/\mathcal{Y}'} \simeq g^* \mathbb{L}_{\mathcal{X}/\mathcal{Y}}$

Sketch: Given
$$\begin{tikzcd} \underline{\mathrm{Spec}\, A} \arrow[r] \arrow[d] & \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X} \arrow[d] \\ \underline{\mathrm{Spec}\, A \oplus M} \arrow[r] & \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$
, lifts $\mathrm{Spec}\, A \oplus M \to \mathcal{X}$ correspond uniquely to lifts $\mathrm{Spec}\, A \oplus M \to \mathcal{X}'$

(Maybe best to just state for $\mathcal{Y}, \mathcal{Y}'$ affine d. schs?)

② We've seen that $\mathbb{L}$ controls lifts of split square-0 extensions $\operatorname{Spec} A \longrightarrow \operatorname{Spec} A \oplus M$

2 ways to generalize:

- Globalize to split square-0 exts $\mathcal{Z} \to \mathcal{Z}[F]$, $F \in QCoh(\mathcal{Z})^{\leq 0}$
- Discuss more general exts $\operatorname{Spec} A \longrightarrow \operatorname{Spec} A \oplus_d M[-1]$

The former is straightforward: If $F \in QCoh(\mathcal{Z})^{\leq 0}$, define
$\mathcal{Z}(F)(A) = \{(a: \operatorname{Spec} A \to \mathcal{Z},\ s: \operatorname{Spec} A \to \operatorname{Spec} A \oplus a^*F \text{ section})\}$
or equivalently $\mathcal{Z}(F) = \operatorname{colim}_{a: \operatorname{Spec} A \to \mathcal{Z}} \operatorname{Spec} A \oplus a^*F$

Prop: Let $\mathcal{X}, \mathcal{Y} : \underline{dComm_k} \longrightarrow \underline{\mathcal{S}}$ be d. stks s.t. $\mathcal{Y}$ admits cotan. cx.
Given $f: \mathcal{X} \to \mathcal{Y}$, have iso (nat. in $F \in QCoh(\mathcal{X})^{\leq 0}$)
$\operatorname{Map}_k(\mathcal{X}[F], \mathcal{Y}) \times_{\operatorname{Map}_k(\mathcal{X},\mathcal{Y}), f} pt \simeq \Omega^\infty \operatorname{Hom}_{\mathcal{X}}(f^*\mathbb{L}_{\mathcal{Y}}, F)$

Pf: Write both sides as limits over all $a: \operatorname{Spec} A \to \mathcal{X}$

This is maybe a bit quotidian, but we can use it to compute $\mathbb{L}_{\underline{\operatorname{Map}}_k(\mathcal{X},\mathcal{Y})}$

Recall: $\underline{\operatorname{Map}}_k(\mathcal{X},\mathcal{Y})(A) = \operatorname{Map}_k(\mathcal{X}_A, \mathcal{Y})$

In nice cases (e.g. $\mathcal{X}$ sch flat + proper /k, $\mathcal{Y}$ derived Artin loc. fp /k)
this is derived Artin, loc. fp /k (by Artin-Lurie representability - details?)

Prop: If X sch flat + proper /k, then $f^*: QCoh(\operatorname{Spec} k) \to QCoh(X)$
admits left adj given on $Perf(X)$ by $F \mapsto (f_*(F^\vee))^\vee$

Prop: In "nice case" above, have $\boxed{\mathbb{L}_{\underline{\operatorname{Map}_k}(X,\mathcal{Y})} = (\pi_*(ev^*\mathbb{L}_{\mathcal{Y}}^\vee))^\vee}$

Pf: For $f: X_A \to \mathcal{Y}$, have $\underline{\operatorname{Map}}_k(X,\mathcal{Y})(A\oplus M) \times_{\underline{\operatorname{Map}}(X,\mathcal{Y})(A), f} pt \simeq \Omega^\infty \operatorname{Hom}_{X_A}(f^*\mathbb{L}_{\mathcal{Y}}, \pi_A^* M)$
$\qquad \wr\!\wr \quad \Omega^\infty \operatorname{Hom}_A((\pi_{A*}(f^*\mathbb{L}_{\mathcal{Y}}^\vee))^\vee, M)$
where $\pi_A: X_A \to \operatorname{Spec} A$ (since $X_{A\oplus M} = X_A[\pi_A^* M]$).
Using above adjunction + currying, get result.

This can be shown more generally - cf. Halpern-Leistner & Preygel

③ Obstruction thy & non-split lifting problems

This discussion can be made global, but let's focus on the local story for simplicity

We really want to consider lifting of general square-zero extensions

For $A \in \underline{d\mathrm{Comm}}$, $M \in \mathrm{Mod}_A^{\le 0}$ (note!), $d : A \to M$ a k-derivation, let

$A \oplus_d M[-1] = A \times_{(id,0),\, A \oplus M,\, (id,d)} A$ = "$\{(a \in A,\ da \sim 0)\}$"

$$\begin{tikzcd} A \oplus_d M[-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "(id,d)"] \\ A \arrow[r, "(id,0)"] & A \oplus M \end{tikzcd}$$

Then we have $A \oplus_d M[-1] \to A$, but not reverse in general

Say a d. stk $\mathcal{X}$ has an obstruction thy if $\mathbb{L}_{\mathcal{X}}$ exists & $\mathcal{X}$ preserves these fiber squares "$\mathcal{X}$ is infinitesimally Cartesian"
(can also relativize this ... we'll skip)

Prop: Every derived Artin stack has an obst. thy

Prop: Given lifting problem

$$\begin{tikzcd} \mathrm{Spec}\, A \arrow[r, "x"] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{Spec}\, A \oplus_d M[-1] \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}\, k \end{tikzcd}$$

where $\mathcal{X}$ has obst. thy

i) $\exists$ canonical obstruction class $\alpha \in \pi_0 \Omega^\infty \mathrm{Hom}_A(x^* \mathbb{L}_{\mathcal{X}}, M)$ st
$\alpha = 0 \iff$ lift exists

ii) When $\alpha = 0$, space of lifts is a $\Omega^\infty \mathrm{Hom}_A(x^* \mathbb{L}_{\mathcal{X}}, M[-1])$-torsor

Pf: Space of lifts is $\mathcal{X}(A \oplus_d M[-1]) \times_{\mathcal{X}(A), x} pt =: L$

Applying $(-) \times_{\mathcal{X}(A), x} pt$ to

$$\begin{tikzcd} \mathcal{X}(A \oplus_d M[-1]) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X}(A) \arrow[d] \\ \mathcal{X}(A) \arrow[r] & \mathcal{X}(A \oplus M) \end{tikzcd}$$

get

$$\begin{tikzcd} L \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & pt \arrow[d, "d"] \\ pt \arrow[r, "0"] & \Omega^\infty \mathrm{Hom}_A(x^* \mathbb{L}_{\mathcal{X}}, M) \end{tikzcd}$$

Let α = conn. cpnt of image of d, then fiber prod is nonempty iff $\alpha = 0$, in which case get noncan iso $L \simeq \Omega(\Omega^\infty \mathrm{Hom}_A(x^* \mathbb{L}_{\mathcal{X}}, M)) \simeq \Omega^\infty \mathrm{Hom}_A(x^* \mathbb{L}_{\mathcal{X}}, M[-1])$

Application: For Postnikov towers of derived Artin stacks, can write $t_{\le n+1}(\mathcal{X}) = t_{\le n+1}(\mathcal{X})_d[\pi_{n+1}(\mathcal{X})[n+1]]$, giving k-invts & obstruction thy as in algebraic topology

4 How to compute cotangent complex of a groupoid quotient

Let $\cdots\ \mathfrak{X}_1 \;\substack{\xrightarrow{s} \\ \xleftarrow{e} \\ \xrightarrow{t}}\; \mathfrak{X}_0$ be a sm. gpd in derived $(n-1)$-Artin stacks, and let $\mathfrak{X} = |\mathfrak{X}_\bullet|$ be the quotient (a derived n-Artin stack)

Suppose $\pi : \mathfrak{X}_0 \to \mathfrak{X}$ is the quotient map

Conormal fiber seq $\pi^*\mathbb{L}_{\mathfrak{X}} \to \mathbb{L}_{\mathfrak{X}_0} \to \mathbb{L}_{\mathfrak{X}_0/\mathfrak{X}} \to$ on $\mathfrak{X}_0$

Combined w/ $\mathbb{L}_{\mathfrak{X}_0/\mathfrak{X}} = e^*s^*\mathbb{L}_{\mathfrak{X}_0/\mathfrak{X}} = e^*\mathbb{L}_{\mathfrak{X}_1/\mathfrak{X}_0}$

$$\left\{ \begin{tikzcd} \mathfrak{X}_1 \arrow[r, "s"] \arrow[d, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{X}_0 \arrow[d, "\pi"] \\ \mathfrak{X}_0 \arrow[r, "\pi"] & \mathfrak{X} \end{tikzcd} \right.$$

gives $\boxed{\pi^*\mathbb{L}_{\mathfrak{X}} = \mathrm{fib}(\mathbb{L}_{\mathfrak{X}_0} \to e^*\mathbb{L}_{\mathfrak{X}_1/\mathfrak{X}_0})}$

Things we can do with this:

- Remembering descent data lets us construct $\mathbb{L}_{\mathfrak{X}}$ from $\pi^*\mathbb{L}_{\mathfrak{X}}$
  $\Rightarrow$ inductive construction of cotan. cxes
- By induction, each $\mathbb{L}_{\mathfrak{X}_i}$ has amp. in $(-\infty, n-1]$
  $\Rightarrow \pi^*\mathbb{L}_{\mathfrak{X}}$ and $\mathbb{L}_{\mathfrak{X}}$ have amp. in $(-\infty, n]$
- Explicit computations!

Ex: G sm. alg. gp, $\pi : pt \to BG = [pt/G]$

Then $\pi^*\mathbb{L}_{BG} = \mathrm{fib}(\mathbb{L}_{pt} \to e^*\mathbb{L}_{G/pt}) = \mathrm{fib}(0 \to \mathfrak{g}^\vee) = \mathfrak{g}^\vee[-1]$

Ex: More generally, if X sm sch, $G \curvearrowright X$, and $\pi : X \to [X/G]$, then

$\pi^*\mathbb{L}_{[X/G]} = \mathrm{fib}(\mathbb{L}_X \to e^*\mathbb{L}_{G\times X/X}) = \mathrm{fib}(\Omega^1_X \to \mathfrak{g}^\vee \otimes \mathcal{O}_X)$

Ex: Alternatively, if A sm. ab. alg. gp, $\pi : pt \to B^nA$,

then $\pi^*\mathbb{L}_{B^nA} = \mathrm{fib}(\mathbb{L}_{pt} \to e^*\mathbb{L}_{B^{n-1}A}) = \mathrm{fib}(0 \to \mathfrak{a}^\vee[-(n-1)]) = \mathfrak{a}^\vee[-n]$