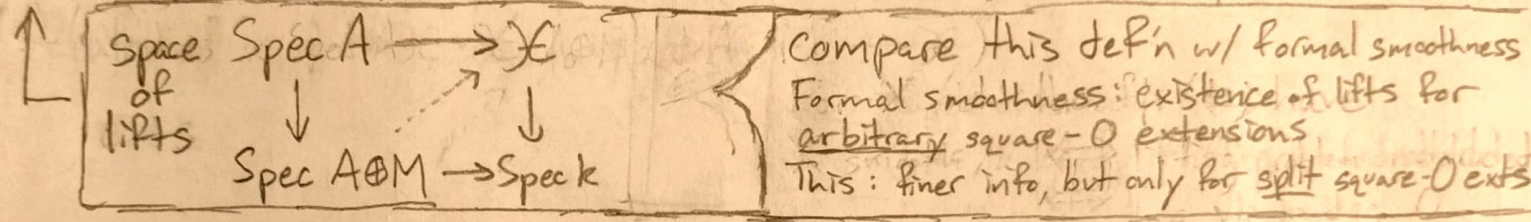


① Some things about cotangent complexes it might be good to know

Recall: A derived comm. ring, $M \in \text{Mod}_A^{\leq 0} \Rightarrow$ get algebra structure on $A \oplus M$ via usual mult on A , $A \otimes M$, and $m^2=0$ for $m \in M$

For $\mathcal{X} : \underline{\text{dComm}}_k \rightarrow \underline{S}$ a derived stack, the (absolute) cotangent complex of \mathcal{X} is $L_{\mathcal{X}} \in D_{qc}(\mathcal{X})$ satisfying $(\forall A, M)$
 $\mathcal{X}(A \oplus M) \times_{\mathcal{X}(A), x} \text{pt} \simeq \Omega^\infty \text{Hom}_A(x^* L_{\mathcal{X}}, M)$ ↳ Ω^∞ is just turning RHS from cx. into space



Prop: For \mathcal{X} derived Artin stack, $L_{\mathcal{X}}$ exists & is unique

Can also relativize: for $f: \mathcal{X} \rightarrow \mathcal{Y}$, have $f^* L_{\mathcal{Y}} \rightarrow L_{\mathcal{X}} \rightarrow L_{\mathcal{X}/\mathcal{Y}} \rightarrow$

More generally: $\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ & \searrow & \downarrow \\ & & \mathcal{Z} \end{array}$ gives $f^* L_{\mathcal{Y}/\mathcal{Z}} \rightarrow L_{\mathcal{X}/\mathcal{Y}} \rightarrow L_{\mathcal{X}/\mathcal{Z}} \rightarrow$
 (follows from above)

Prop: For Cartesian $\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g} & \mathcal{X} \\ \downarrow p' & \lrcorner & \downarrow p \\ \mathcal{Y}' & \xrightarrow{f} & \mathcal{Y} \end{array}$ have $L_{\mathcal{X}'/\mathcal{Y}'} \simeq g^* L_{\mathcal{X}/\mathcal{Y}}$

Sketch: Given $\begin{array}{ccc} \text{Spec } A & \longrightarrow & \mathcal{X}' \longrightarrow \mathcal{X} \\ \downarrow & & \downarrow \lrcorner \downarrow \\ \text{Spec } A \oplus M & \longrightarrow & \mathcal{Y}' \longrightarrow \mathcal{Y} \end{array}$ lifts $\text{Spec } A \oplus M \rightarrow \mathcal{X}$ correspond uniquely to lifts $\text{Spec } A \oplus M \rightarrow \mathcal{X}'$

(Maybe best to just state for $\mathcal{Y}, \mathcal{Y}'$ affine d. schs?)

② We've seen that \mathbb{L} controls lifts of split square-0 extensions $\text{Spec } A \rightarrow \text{Spec } A \oplus M$

2 ways to generalize:

- Globalize to split square-0 exts $\mathbb{X} \rightarrow \mathbb{X}[F], F \in \text{QCoh}(\mathbb{X})^{\leq 0}$
- Discuss more general exts $\text{Spec } A \rightarrow \text{Spec } A \oplus M[-1]$

The former is straightforward: If $F \in \text{QCoh}(\mathbb{X})^{\leq 0}$, define $\mathbb{X}(F)(A) = \{(a: \text{Spec } A \rightarrow \mathbb{X}, s: \text{Spec } A \rightarrow \text{Spec } A \oplus a^*F \text{ section})\}$ or equivalently $\mathbb{X}(F) = \text{colim}_{a: \text{Spec } A \rightarrow \mathbb{X}} \text{Spec } A \oplus a^*F$

Prop: Let $\mathbb{X}, \mathbb{Y}: \underline{\text{dComm}}_k \rightarrow \underline{\mathcal{S}}$ be d.stks s.t. \mathbb{Y} admits cotan. cx.

Given $f: \mathbb{X} \rightarrow \mathbb{Y}$, have iso (nat. in $F \in \text{QCoh}(\mathbb{X})^{\leq 0}$)

$$\text{Map}_k(\mathbb{X}[F], \mathbb{Y})_{\text{Map}_k(\mathbb{X}, \mathbb{Y}), f}^{\times} \text{pt} \simeq \Omega^{\infty} \text{Hom}_{\mathbb{X}}(f^* \mathbb{L}_{\mathbb{Y}}, F)$$

Pf: Write both sides as limits over all $a: \text{Spec } A \rightarrow \mathbb{X}$

This is maybe a bit quotidian, but we can use it to compute $\mathbb{L}_{\text{Map}_k(\mathbb{X}, \mathbb{Y})}$

Recall: $\text{Map}_k(\mathbb{X}, \mathbb{Y})(A) = \text{Map}_k(\mathbb{X}_A, \mathbb{Y})$

In nice cases (e.g. \mathbb{X} sch flat+proper/k, \mathbb{Y} derived Artin loc. fp/k) this is derived Artin, loc. fp/k (by Artin-Lurie representability - details?)

Prop: If X sch flat+proper/k, then $f^*: \text{QCoh}(\text{Spec } k) \rightarrow \text{QCoh}(X)$ admits left adj given on $\text{Perf}(X)$ by $F \mapsto (f_*(F^\vee))^\vee$

Prop: In "nice case" above, have $\mathbb{L}_{\text{Map}_k(X, \mathbb{Y})} = (\pi_*(\text{ev}^* \mathbb{L}_{\mathbb{Y}}^\vee))^\vee$

Pf: For $f: X_A \rightarrow \mathbb{Y}$, have $\text{Map}_k(X, \mathbb{Y})(A \oplus M)_{\text{Map}_k(X, \mathbb{Y})(A), f}^{\times} \text{pt} \simeq \Omega^{\infty} \text{Hom}_{X_A}(f^* \mathbb{L}_{\mathbb{Y}}, \pi_A^* M)$ where $\pi_A: X_A \rightarrow \text{Spec } A$ (since $X_{A \oplus M} = X_A[\pi_A^* M]$).

Using above adjunction+currying, get result.

$$\Omega^{\infty} \text{Hom}_A((\pi_A^*(f^* \mathbb{L}_{\mathbb{Y}}^\vee))^\vee, M)$$

This can be shown more generally - cf. Halpern-Leistner & Preygel

③ Obstruction thy & non-split lifting problems

This discussion can be made global, but let's focus on the local story for simplicity

We really want to consider lifting of general square-zero extensions

For $A \in \underline{dComm}$, $M \in \text{Mod}_A^{\langle 0 \rangle}$ ^{note!}, $d: A \rightarrow M$ a k -derivation, let

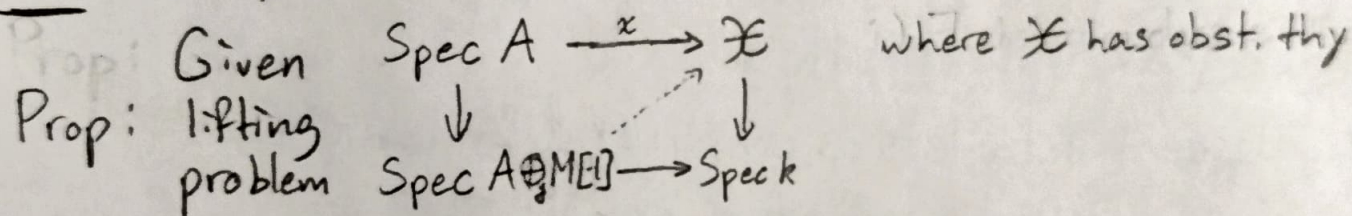
$$A \oplus_d M[-1] = A_{(id,0)} \times_{A \oplus M, (id,d)} A = \{(a \in A, da \sim 0)\}$$

$$\begin{cases} A \oplus_d M[-1] \rightarrow A \\ \downarrow \quad \quad \quad \downarrow \\ A \xrightarrow{(id,0)} A \oplus M \end{cases}$$

Then we have $A \oplus_d M[-1] \rightarrow A$, but not reverse in general

Say a d. stk \mathcal{X} has an obstruction thy if $L_{\mathcal{X}}$ exists & \mathcal{X} preserves these fiber squares
 (can also relativize this ... we'll skip) " \mathcal{X} is infinitesimally Cartesian"

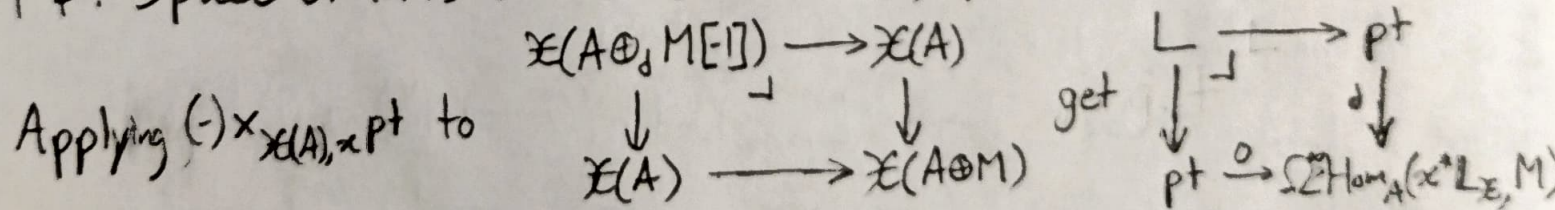
Prop: Every derived Artin stack has an obst. thy



i) \exists canonical obstruction class $\alpha \in \pi_0 \Omega^{\infty} \text{Hom}_A(x^* L_{\mathcal{X}}, M)$ st $\alpha = 0 \iff$ lift exists

ii) When $\alpha = 0$, space of lifts is a $\Omega^{\infty} \text{Hom}_A(x^* L_{\mathcal{X}}, M[-1])$ -torsor

PF: Space of lifts is $\mathcal{X}(A \oplus_d M[-1]) \times_{\mathcal{X}(A), x} \text{pt} =: L$



Let $\alpha = \text{conn. cpnt of image of } d$, then fiber prod is nonempty iff $\alpha = 0$, in which case get noncan iso $L \simeq \Omega(\Omega^{\infty} \text{Hom}_A(x^* L_{\mathcal{X}}, M)) \simeq \Omega^{\infty} \text{Hom}_A(x^* L_{\mathcal{X}}, M[-1])$

Application: For Postnikov towers of derived Artin stacks, can write $t_{\leq n+1}(\mathcal{X}) = t_{\leq n+1}(\mathcal{X})_d[\pi_{n+1}(\mathcal{X})[n+1]]$, giving k -invs & obstruction thy as in algebraic topology

(4) How to compute cotangent complex of a groupoid quotient

Let $\dots \mathcal{X}_1 \xrightleftharpoons[e]{s} \mathcal{X}_0$ be a sm. gpd in derived $(n-1)$ -Artin stacks,

and let $\mathcal{X} = |\mathcal{X}_\bullet|$ be the quotient (a derived n -Artin stack)

Suppose $\pi: \mathcal{X}_0 \rightarrow \mathcal{X}$ is the quotient map

Conormal fiber seq $\pi^* \mathbb{L}_{\mathcal{X}} \rightarrow \mathbb{L}_{\mathcal{X}_0} \rightarrow \mathbb{L}_{\mathcal{X}_0/\mathcal{X}} \rightarrow 0$ on \mathcal{X}_0

Combined w/ $\mathbb{L}_{\mathcal{X}_0/\mathcal{X}} = e^* s^* \mathbb{L}_{\mathcal{X}_0/\mathcal{X}} = e^* \mathbb{L}_{\mathcal{X}_1/\mathcal{X}_0}$

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{s} & \mathcal{X}_0 \\ \downarrow e & & \downarrow \pi \\ \mathcal{X}_0 & \xrightarrow{\pi} & \mathcal{X} \end{array}$$

gives $\boxed{\pi^* \mathbb{L}_{\mathcal{X}} = \text{fib}(\mathbb{L}_{\mathcal{X}_0} \rightarrow e^* \mathbb{L}_{\mathcal{X}_1/\mathcal{X}_0})}$

Things we can do with this:

- Remembering descent data lets us construct $\mathbb{L}_{\mathcal{X}}$ from $\pi^* \mathbb{L}_{\mathcal{X}}$
 \Rightarrow inductive construction of cotan. complexes
- By induction, each $\mathbb{L}_{\mathcal{X}_i}$ has amp. in $(-\infty, n-1]$
 $\Rightarrow \pi^* \mathbb{L}_{\mathcal{X}}$ and $\mathbb{L}_{\mathcal{X}}$ have amp. in $(-\infty, n]$
- Explicit computations!

Ex: G sm. alg. gp, $\pi: \text{pt} \rightarrow BG = [\text{pt}/G]$

Then $\pi^* \mathbb{L}_{BG} = \text{fib}(\mathbb{L}_{\text{pt}} \rightarrow e^* \mathbb{L}_{G/\text{pt}}) = \text{fib}(0 \rightarrow \mathfrak{g}^\vee) = \mathfrak{g}^\vee[-1]$

Ex: More generally, if X sm sch, $G \curvearrowright X$, and $\pi: X \rightarrow [X/G]$, then

$\pi^* \mathbb{L}_{[X/G]} = \text{fib}(\mathbb{L}_X \rightarrow e^* \mathbb{L}_{G \cdot X/X}) = \text{fib}(\Omega'_X \rightarrow \mathfrak{g}^\vee \otimes \theta_X)$

Ex: Alternatively, if A sm. ab. alg. gp, $\pi: \text{pt} \rightarrow B^n A$,

then $\pi^* \mathbb{L}_{B^n A} = \text{fib}(\mathbb{L}_{\text{pt}} \rightarrow e^* \mathbb{L}_{B^n A}) = \text{fib}(0 \rightarrow \mathfrak{a}^\vee[-(n-1)]) = \mathfrak{a}^\vee[-n]$