Analytic Langlands from 4D Super Yang-Mills

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1 Introduction

Geometric Langlands has a natural physical setting as a twist of four dimensional, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM). An easy way to obtain this theory is by compactification (i.e. dimensional reduction) of the essentially unique ten dimensional SYM (which is the maximal allowed dimension for SYM). This is discussed in some detail in [7]. We then topologically twist this theory to obtain a topological field theory (TFT), see [7] for more details. The TFT has a partition function which to each four-manifold assigns a number, to each three-manifold assigns a Hilbert space, and to each two-manifold assigns a category of branes, or boundary conditions. The categorical Langlands correspondence is a correspondence between the categories assigned to two (S-dual) two-manifolds.

Recently, Etingof-Frenkel-Kazhdan have discovered an analytic version of the Langlands correspondence, which instead of functors and categories one instead has Hilbert spaces and self-adjoint operators as the duality parameters. This year, Gaiotto-Witten reinterpreted this story in a gauge-theoretic setting [5]. The purpose of this note is to describe some of the basic features of this setting, how it encompasses the discovered analytic Langlands duality, and how physical dualities and notions imply some of the unproven conjectures of Etingof-Frenkel-Kazhdan.

2 Review of Categorical Geometric Langlands in SYM

2.1 Review of Four-dimensional SYM

Four-dimensional $\mathcal{N} = 4$ SYM is most easily obtained from dimensional reduction of $\mathcal{N} = 4$ ten-dimensional SYM. In the ten-dimensional theory, the fields are the gauge field A, i.e. a connection on a G-bundle E, and a fermion field λ that is a section of $\mathcal{S}^+ \otimes \mathrm{ad}(E)$, where $\mathcal{S}^+$ is the (positive) spin representation of $\mathrm{SO}(10)$. We have the covariant derivative $D = \mathrm{d}+A$ where d is the de Rham differential, and the curvature of A is $F = \mathrm{d}A + A \wedge A$. Reduction to four dimensions is then quite simple: Take all fields to be independent of the coordinates $x^4, \dots, x^9$. Thus, the other 6 components of the gauge field A_{i+4} define new four-dimensional scalar fields ϕ_i, $i \in \{0, \dots, 5\}$. Similarly, the fermion field defines new four-dimensional fermionic fields ψ_i. The ten-dimensional action then has a neat expression as a four-dimensional one, which we will not reproduce here.

We now twist the theory thus obtained to obtain a TFT on any suitable four-manifold M. A detailed description of the twisting procedure can be found in [7], and it's not especially relevant for the current paper, so we will omit it.

2.2 Categorical Geometric Langlands in the Context of SYM

In the mathematical theory of geometric Langlands, one begins with a smooth, projective curve $X/\mathbb{C}$ and a complex reductive algebraic group G. Then one has a correspondence between the moduli space of flat, holomorphic line bundles on X with connection and Hecke eigensheaves on $\mathrm{Bun}_G(X)$. Physically, we study the twisted SYM on $M = \Sigma \times X$, and compactify on X to obtain a two-dimensional sigma-model on Σ. This model is of maps $\Sigma \to \mathcal{M}_H(X, G)$, where $\mathcal{M}_H(X, G)$ is the Hitchin moduli space on X with gauge group G, where G is now compact. The original G of geometric Langlands is then $G_{\mathbb{C}}$, the complexification of this compact G. Explicitly, $\mathcal{M}_H(X, G)$ is a complex manifold of dimension $(2g-2)\dim G$. It is the space of solutions to Hitchin's equations for the group G on X. It is the moduli space of G Higgs bundles on X, where a G Higgs bundle is just a principal G-bundle E along with a Higgs field, φ, which is a holomorphic section of $K_X \otimes \mathrm{ad}\, E$. The connection to geometric Langlands is that Wilson and 't Hooft operators in the four-dimensional theory reduce to line operators on Σ that can be thought of as functors on the category of D-branes (Dirichlet boundary conditions) of the sigma-model. These functors have "eigenbranes," and these are the Hecke eigensheaves of the geometric Langlands correspondence. The line operators in question are then the Hecke functors.

One can view usual geometric Langlands as a deformation quantization of the algebra of *holomorphic* functions on $\mathcal{M}_H(X, G)$. On the other hand, the analytic theory can be obtained by geometrically quantizing this real symplectic manifold. Thus, a certain class of functions (not necessarily holomorphic or antiholomorphic) becomes a class of operators acting on a certain Hilbert space. Upon working through this task, we find that S-duality of boundary conditions naturally encodes the oper condition. Moreover, we find that certain conjectures of Etingof-Frenkel-Kazhdan are answered in the positive by the physics.

3 Geometric Quantization of $\mathcal{M}_H(X, G)$

3.1 Brane Quantization

Before discussing how to quantize the sigma-model discussed above, we will first make some general remarks about brane quantization, which was discussed in detail in [Gukov-Witten]. Let Y be a complex symplectic manifold with complex structure I and holomorphic symplectic form Ω. We can view Y as a real symplectic manifold with the real symplectic form $\omega = \mathrm{Im}\,\Omega$. Supposing that a σ-model with target Y exists, which is believed to be true if Ω can be extended to a complete hyper-Kähler structure, we can twist this σ-model to an A-model. The A-model of Y can have in addition to the usual lagrangian branes also coisotropic branes. The simplest possible case is a rank 1 coisotropic A-brane supported on all of Y.[1] Let B be the B-field of the σ-model, which is just a 2-form field. It actually doesn't matter what this map is or does, as we will soon set it to 0. Further on Y, we can consider a brane with support all of Y whose Chan-Paton (CP) line bundle has a unitary connection A with curvature $F = \mathrm{d}A$. A CP line bundle on a brane is one which paramatrizes the CP factors that can be assigned to its boundary; see [8] for a physical explanation of these factors. For this to be an A-brane, Kapustin-Orlov found that $I' = \omega^{-1}(F + B)$ should be an integrable complex structure on Y. Two possible cases are now obvious: We could choose $F + B = \mathrm{Re}\,\Omega$, so $I' = I$. This brane will be denoted $\mathcal{B}_{\mathrm{cc}}$, and we will call it the canonical coistropic A-brane. Another choice is $F + B = -\mathrm{Re}\,\Omega$, so $I' = -I$. We will denote this A-brane as $\overline{\mathcal{B}}_{\mathrm{cc}}$, the conjugate canonical coisotropic A-brane. Now, for convenience, take $B = 0$, so that $F \in \{\mathrm{Re}\,\Omega, -\mathrm{Re}\,\Omega\}$. This choice is possible since for $\mathcal{M}_H(X, G)$, $\mathrm{Re}\,\Omega$ has trivial cohomology, so there exists a complex line bundle on Y with curvature $\mathrm{Re}\,\Omega$.

The algebras of functions which we could deformation quantize are then $\mathcal{A} = \mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$ and

[1] An A-brane is just a Dirichlet boundary condition for the A-model with target Y.

$\overline{\mathcal{A}} = \mathrm{Hom}(\overline{\mathcal{B}}_{\rm cc}, \overline{\mathcal{B}}_{\rm cc})$, where the power series in $\hbar$ is given by $\Omega = \Omega_0/\hbar$ (we keep Ω_0 fixed). As $\hbar \to 0$, $\mathcal{A} \to \mathcal{A}_0$, the commutative algebra of holomorphic functions on Y. At first order, the noncommutative multiplication in $\mathcal{A}$ is given by the Poisson bracket $\{f, g\} = (\Omega^{-1})^{ij}\partial_i f \partial_j g$.

However, we would like to geometrically quantize, so we want to consider a real symplectic manifold M with symplectic form ω, whose complexification will be Y. In practice, we will set M to be the Hitchin moduli space viewed as a real symplectic manifold and $Y = M \times M$, so in particular $\Omega|_M = \omega$, where Ω is the complex symplectic form on Y. Thus, there is an antiholomorphic involution τ on Y such that $\tau^*\Omega = \overline{\Omega}$; it's given by reversing the factors of M. Now, a choice of a prequantum line bundle $\mathfrak{L} \to M$ determines an A-brane $\mathcal{B}$ with support all of M. After we geometrically quantize, the A-model Hilbert space obtained with these inputs is $\mathcal{H} = \mathrm{Hom}(\mathcal{B}, \mathcal{B}_{\rm cc})$. We will now define a hermitian inner product on $\mathcal{H}$.

3.2 Defining the Inner Product

In order to define the inner product, we need to understand the TFT structure of the twisted σ-model on Y. Since $Y = M \times M$ is a product, there are different ways we can interpret branes on Y. In particular, given a brane $\mathcal{B}$, we can consider the space $\mathrm{Hom}(\mathcal{B}, \widehat{\mathcal{B}}_{\rm cc})$, where $\widehat{\mathcal{B}}_{\rm cc}$ is the product canonical coisotropic brane on Y, i.e. $\widehat{\mathcal{B}}_{\rm cc} = \mathcal{B}_{\rm cc} \times \overline{\mathcal{B}}_{\rm cc}$, where each $\mathcal{B}_{\rm cc}$ sits on its copy of M. Explicitly, what we're doing is considering two copies of the σ-model, one on each copy of M. For τ to be an *antiholomorphic* involution, we need the complex structure on one M to be I and on the other to be $-I$. The brane $\mathcal{B}$ is then considered as living on the boundary of the two copies of M, see Fig. 1. Thus, the Hilbert space we're interestedin is $\mathcal{H} = \mathrm{Hom}(\mathcal{B}, \hat{\mathcal{B}}_{\rm cc})$. The TFT structure gives us a nondgenerate bilinear pairing between the dual space $\mathcal{H}^* = \mathrm{Hom}(\hat{\mathcal{B}}_{\rm cc}, \mathcal{B})$, but it isn't necessarily hermitian. To make it hermitian, we need to compose it with an antilinear map $\mathcal{H} \to \mathcal{H}^*$. Physics immediately gives us such a mapping: The CPT symmetry Θ. However, Θ is not an A-model symmetry; rather, it maps the A-model to a copy of itself with opposite symplectic form. Equivalently, it maps the A-model with BRST differential Q to a conjugate A-model with differential $Q^\dagger$. However, the symmetry τ of the σ-model discussed above reverses the sign of the symplectic form; equivalently, it maps Q back to $Q^\dagger$ (and vice-versa). Thus, using the composition $\Theta \circ \tau$ which is self-adjoint antilinear, we get an antilinear symmetry of the A-model. Define then the hermitian inner product to be

$$\langle \psi, \psi' \rangle = (\Theta \circ \tau\psi, \psi'),$$

where $(\cdot, \cdot)$ is the pairing described above between $\mathcal{H}$ and $\mathcal{H}^*$. Now, this pairing is not positive definite in general; however, if we specialize to the case of quantizing the cotangent bundle of the Hitchin moduli space (which we will soon do), which is a dense open contained in $M \times M$, then the pairing is positive definite and is the required inner product. We remark that the proof that the inner product is positive definite (in the physics context) is actually quite subtle, and we do not have the space to include it here. See Appendix B of [5].

3.3 Quantization of $\mathcal{M}_H(X, G)$

In §3.2, we were only considering the reduced theory. Now that we have set out to quantize, we actually want to consider the full four-dimensional theory on $\Sigma \times X$, where we think of Σ as the "unfolded" strip of Fig. 1. We will now digress to explain what this means.

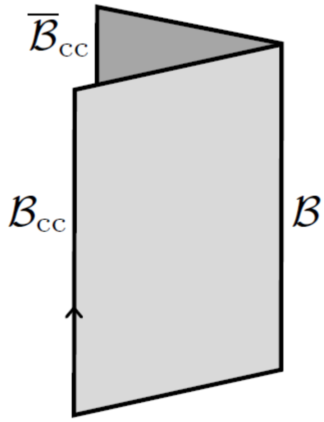

Figure 1: The "folded" σ-model on Y. There are two copies of the Hitchin moduli space M with a boundary condition $\mathcal{B}$ separating them. They each have their own canonical coisotropic branes $\mathcal{B}_{\rm cc}$ and $\overline{\mathcal{B}}_{\rm cc}$, respectively. The reason for the conjugate brane is because the two M's have opposite complex structures.

As was mentioned in §3.2, there are alternative ways to view the σ-model on Y. An alternative description to the one presented in the previous section is to begin with the situation depicted in Fig. 1, and to "unfold" the picture. This means we reverse the orientation of one of the two sheets and set $\mathcal{B}$ to be the trivial boundary condition (i.e. the unit object in the (symmetric monoidal) category of branes). We thus end up with a *single* σ-model on a strip: Before unfolding, the symplectic forms on the two copies of M differed by a sign; reversing the orientation sets them equal. Clearly, in the unfolded picture, $\mathcal{H}$ becomes $\mathrm{Hom}(\overline{\mathcal{B}}_{\rm cc}, \mathcal{B}_{\rm cc})$, and we could define an inner product for it following exactly the same approach as that described in §3.2.

Returning to quantization, a Higgs bundle on X can as usual be described by a pair (A, φ), where A is a connection on the underlying G-bundle $E \to X$ and φ is a section of $K_X \otimes \mathrm{ad}(E)$. Using this description, we can obtain $\overline{\mathcal{B}}_{\rm cc}$ from $\mathcal{B}_{\rm cc}$ via the mapping $(A, \varphi) \mapsto (A, -\varphi)$. Now, note that a dense open set in the Hitchin moduli space $\mathcal{M}_H(X, G)$ is the cotangent bundle $T^*\mathcal{M}(X, G)$, where $\mathcal{M}$ denotes the moduli space of *semistable* holomorphic G-bundles on X. This situation is ideal for geometric quantization, since quantization of cotangent bundles is particualrly simple. Namely, the Hilbert space that is associated to $\mathcal{M}(X, G)$ is just $L^2_{\rm hd}(\mathcal{M}(X, G))$, where $L^2_{\rm hd}$ denotes the space of L^2 half-densities, i.e. the space of L^2 sections of $K^{1/2} \otimes \overline{K}^{1/2}$, where K is the canonical bundle of $\mathcal{M}(X, G)$. Note that since this cotangent bundle is dense in the Hitchin moduli space, we might expect their associated quantum Hilbert spaces to be the same. In fact they are, via a construction in [Brane Quant].

3.4 Operators on the Quantum Hilbert Space

3.4.1 Hitchin Hamiltonians

Now that we have a Hilbert space, we need to identify the classical functions that quantize to operators acting on that space. In analytic Langlands, these are the Hecke operators. Given a Higgs bundle (A, φ), which we remind is a solution of Hitchin's equations, we can consider only the holomorphic (1,0) part of φ, denoted ψ. Hitchin's equations give $\bar{\partial}_A \varphi = 0$, where $\bar{\partial}_A$ is the anitholomorphic Dolbeault differential twisted by the connection A. Given an invariant polynomial on the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of the complexification $G_{\mathbb{C}}$ of G, homogeneous of degree s, then $P(\psi) \in \Gamma(X, K_X^s)$ is holomorphic. Setting $T_X = K_X^{-1}$, and given any $\alpha \in \Gamma(X, \Omega_X^{0,1} \otimes T_X^{s-1})$, i.e. any 1-form on X with values in T_X^{s-1}, we define the **Hitchin hamiltonian** associated to P and α,

$$H_{P,\alpha} = \int_X \alpha P(\psi). \tag{1}$$

This is a holomorphic function on $\mathcal{M}_H(X, G)$; clearly, it depends only on the cohomology class of α in $H^1(X, T_X^{s-1})$. There are only finitely many α, given by the dimension of this space, which is $(s+1)(g-1)$. For example, if $G = \mathrm{SU}(2)$, the ring of holomorphic functions on the Hitchin moduli space is generated by $H_{P,\alpha}$, where $P(\psi) = \mathrm{tr}(\psi^2)$ (for this is the only invariant polynomial on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{su}_2 \cong \mathfrak{sl}_2(\mathbb{C})$). Indeed, given simple G of rank r, we get r independent Casimirs, which in turn give rise to r homogeneous invariant polynomials P_i, and the ring of holomorphic functions on the Hitchin moduli space is generated by the H_{P_i,α_j}. Note that these hamiltonians Poisson-commute, since they are constructed from ψ only. These are the hamiltonians of the classical Hitchin integrable system, and we expect that they will give rise to operators acting on $\mathcal{H}$ after quantization.

For example, deformation quantization of the algebra $\mathcal{A}_0$ of holomorphic functions on $\mathcal{M}_H(X, G)$ to the (generally non-commutative) algebra $\mathcal{A}$ of operators on $\mathcal{H}$ arises by an expansion in $\hbar$, mentioned in §3.1. This deformation is **unobstructed**: For every invariant polynomial P on $\mathfrak{g}_{\mathbb{C}}$ homogeneous of degree s, and for each Hitchin hamiltonian $H_{P,\alpha}$ associated to it, there exists a differential operator $D_{P,\alpha}$ whose leading symbol is $H_{P,\alpha}$ acting on $\Gamma(\mathcal{M}(X, G), K^{1/2})$, where K is the canonical bundle of $\mathcal{M}(X, G)$. Mathematically, this follows from the fact that for any simple G, $H^1(\mathcal{M}(X, G), \mathcal{O}) = 0$. Physically, this follows from the fact that the BRST cohomology of boundary local operators for $\mathcal{B}_{\mathrm{cc}}$ is generated by the operators P and a second type of generator which also commutes with P. The quantum BRST cohomology is then the cohomology of the quantum BRST differential Q which acts on the classical cohomology; however, it turns out that this differential is identically 0. This implies that the quantum cohomology is equal to the classical one; hence, the commuting Hitchin hamiltonians quantize to commuting holomorphic differential operators. In particular, there is no obstruction to the deformation quantization of $\mathcal{A}_0$. In the next subsection we will explain this gauge theory picture.

3.4.2 Quantization of the Hitchin Hamiltonians

[K-W] To see what the boundary local operators for $\mathcal{B}_{\mathrm{cc}}$ are, it is essential to pass to the four-dimensional picture. Consider local operators $\mathcal{O}$ inserted at a point p of the boundary of $\Sigma \times X$, which we note looks like $\mathbb{R} \times X$. Consider the cohomology of the topological supercharge Q of the A-model in complex structure K which acts on local operators inserted at $(s, p) \in \mathbb{R} \times X$. Further, note that the metric of X is irrelevant modulo $\{Q, \cdots\}$ (i.e. modulo Q-exact terms), so we may as well assume X flat near p.

Now, $\mathcal{O}$ are classified by their dimension and "spin," i.e. by (half of) the highest weight of the representation under which they transform by rotations of X about p. Thus, consider Q-invariant operators of dimension n and spin also n. A gauge invariant operator must be constructed only from ϕ_z, ψ_z, and $\widetilde{\psi}_z$, and the covariant derivative D_z. These fields are linear combinations of the fields of the four-dimensional theory described in §2.1. The subscript z denotes that we take the holomorphic part of the field, viewed in the complex structure J on the Hitchin moduli space. In other words, we define a local complex parameter z on X and take those parts of the fields that are holomorphic in z; for example, A_z appears in $A = A_z\,\mathrm{d}z + A_{\bar{z}}\,\mathrm{d}\bar{z}$. However, since A_z is not Q-invariant on the boundary, D_z cannot appear. On the boundary, $\psi_z = \widetilde{\psi}_z$, so boundary observables that are Q-invariant of dimension n and spin n are functions just of ϕ_z and ψ_z, without any derivatives. Now, ϕ_z and ψ_z have fermion ghost numbers $\mathcal{K} = 0$ and $\mathcal{K} = 1$, respectively.[2] Thus, an operator with $\mathcal{K} = 0$ is a gauge-invariant function of ϕ_z only, e.g. $\operatorname{tr}\phi_z^n$, while a typical operator of $\mathcal{K} = 1$ is e.g. $\mathcal{O}' = \operatorname{tr}(\phi_z^{n-1}\psi_z)$. Classically, the boundary conditions ensure that $[Q, \phi_z] = 0$ and $\{Q, \psi_z\} = 0$, so all boundary local operators $\mathcal{O}$, $\mathcal{O}'$ are nonzero elements of the cohomology of Q. What are the quantum corrections to this result? For example, could there be a correction such that $[Q, \mathcal{O}] = \varepsilon\mathcal{O}'$ for some $\varepsilon \neq 0$? This would imply that a linear combination of $\mathcal{O}$ and $\mathcal{O}'$ now lives in the cohomology, while $\mathcal{O}$ and $\mathcal{O}'$ disappear from it.

Time-reversal symmetry $\mathcal{T}$ shows that such a quantum correction is impossible. This is an orientation-reversing symmetry on the spacetime manifold which changes the sign of the time coordinate $x^0 \mapsto -x_0$ and leaves the other coordinates unchanged. Now, various formulas in [7], namely Equations 3.56-3.58, imply that $\mathcal{T}$ is a symmetry of $\mathcal{B}_{\mathrm{cc}}$, it commutes with the topological supercharge of the A-model in complex structure K, ϕ_z is odd under $\mathcal{T}$, and ψ_z is even under $\mathcal{T}$. Thus, the operators $\mathcal{O}$ and $\mathcal{O}'$ cannot "pair up" under the action of Q in its cohomology, so they remain in the full quantum cohomology. Given an arbitrary boundary local operator of dimension and spin n and ghost number k that is Q-invariant classically, it is determined by a gauge-invariant function $f(\phi_z, \psi_z)$ of degree $n-k$ in ϕ and k in ψ_z; hence, its eigenvalue is $(-1)^{n-k}$ under action by $\mathcal{T}$. But for Q to act on it non-trivially, n would have to remain the same while k would increase by 1; this would switch the sign of the eigenvalue of $\mathcal{T}$, which is impossible. Hence, even these more general operators are preserved under quantization.

Now, by Theorem B in [4] and the discussion in the latter half of Appendix A [5], the classical BRST cohomology is generated by invariant polynomial operators either of the form $P(\phi_z)$ or of the form $P'(\phi_z + \varepsilon\psi_z) = P(\phi_z) + \varepsilon P'(\phi_z, \psi_z)$, where ε is an odd (i.e. Grassmann) parameter, and their z-derivatives. Armed with this result, we'd like to prove that there are no quantum corrections. In fact, the above argument does not cover all possible cases. The Chevalley involution $\mathcal{C}$ of a Lie algebra $\mathfrak{g}$ (which descends to the associated group G) is called charge conjugation by physicists. In the case that it is an inner automorphism of G, then the above analysis of quantum corrections is correct. However, in the event that $\mathcal{C}$ is *outer*, then it is in fact possible to find nontrivial operators such that $[Q, P] = \mathcal{O}$ or $\{Q, P'\} = \mathcal{O}$ that *are* consistent with the $\mathcal{T}$ symmetry. The key is that the cohomology is also generated by z-derivatives of the polynomials in addition to the polynomials themselves.

Now, we can determine whether a given an invariant polynomial P on $\mathfrak{g}$ is even or odd under $\mathcal{C}$ by restricting it to the Lie subalgebra of $\mathfrak{g}$ associated to a maximal torus of G (since this restriction will be nontrivial for nontrivial P). In this case, we have

Lemma 1. For simple G, $\mathcal{C}P(\phi_z)\mathcal{C}^{-1} = P(-\phi_z)$, and $\mathcal{C}P'(\phi_z, \psi_z)\mathcal{C}^{-1} = P'(-\phi_z, -\psi_z) = -P'(-\phi_z, \psi_z)$.

Proof. To say P is an invariant polynomial is to say that it is invariant by the adjoint action of the group G on $\mathfrak{g}$. By definition, $\mathcal{C}$ acts as -1 on every root vector, so we're done, since, again by definition, ϕ and ψ are valued in the adjoint representation of G. □

Applying this lemma to the result of the analysis of the $\mathcal{T}$ symmetry, we find that $\mathcal{C}\mathcal{T}P(\phi_z)(\mathcal{C}\mathcal{T})^{-1} = P(\phi_z)$ and $\mathcal{C}\mathcal{T}P'(\phi_z, \psi_z)(\mathcal{C}\mathcal{T})^{-1} = -P'(\phi_z, \psi_z)$. Thus, the composite operator $\mathcal{C}\mathcal{T}$ acts as $(-1)^k$ on the part of the classical cohomology that is degree k in the operators P' and their derivatives. Since the BRST differential commutes with $\mathcal{C}\mathcal{T}$, this operator must act trivially on the classical cohomology, since the left and right hand sides of a formula $[Q, P] = \mathcal{O}$ or $[Q, P'] = \mathcal{O}$ would transform oppositely under $\mathcal{C}\mathcal{T}$.

[2]Ghost numbers in the A-model are just the degree or dimension of the cohomology class of the field. They actually arise from a symmetry of the topological twist of four-dimensional SYM. For more on this, see §3.1 of [7].

Now, we claim the following.

Proposition 1. The algebra $\mathcal{A}$ is commutative.

Physics Proof. The definition of $H_{P,\alpha}$ given in Eq. 1 depends only on the cohomology class of α in $H^1(X, T_X^{s-1})$. We can thus choose a representative of such a class with support in an arbitrarily small open ball in X. Thus, given $a, a' \in \mathcal{A}$, we can assume that they are represented by operators with disjoint support in X; hence, when represented as living on the boundary of Σ pictured in Fig. 2, they can "slide" past each other in the TFT. This is equivalent to them commuting. □

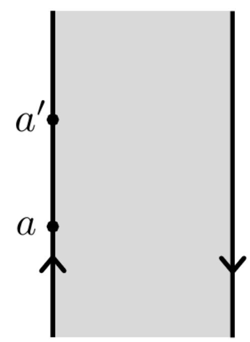

Figure 2: In general, in two-dimensional TFT, $\mathcal{A}$ can be non-commutative, since $a, a' \in \mathcal{A}$ can be inserted on the boundary in a particular order. The ability to slide them past each other in the TFT without singularity is equivalent to commutativity of a, a'. However, for the TFT studied here, there are two additional dimensions not pictured which correspond to the dimensions of X (only Σ and its boundary is pictured). Assuming that a and a' have disjoint support in X, they can be moved past each other without singularity.

We recall that brane quantization of $\mathcal{M}_H(X, G)$ is equivalent to quantizing $T^*\mathcal{M}(X, G)$. In geometric quantization of any cotangent bundle, a function whose restriction to a fiber of the bundle is a homogeneous polynomial of degree s becomes a differential operator, also of degree s, which acts on half-densities on the base of the bundle. From the holomorphic point of view, holomorphic functions on $T^*\mathcal{M}(X, G)$ become holomorphic differential operators acting on sections of $K^{1/2}$. Since any holomorphic differential operator has trivial action on sections of $\overline{K}^{1/2}$, such an operator naturally acts on $K^{1/2} \otimes \overline{K}^{1/2}$. Of course, the same story holds for antiholomorphic functions. Therefore, $\mathcal{A}$ and $\overline{\mathcal{A}}$ become algebras of holomorphic and antiholomorphic differential operators acting on $K^{1/2} \otimes \overline{K}^{1/2}$, respectively; clearly, they commute.

3.4.3 Hecke, 't Hooft, and Wilson Operators

In categorical geometric Langlands, we have Hecke functors which act on the category of A-branes and the "eigenbranes," or Hecke eigensheaves, of these Hecke functors. This picture was put in a gauge-theoretic framework in [7], where the Hecke functors are 't Hooft line operators. Electric-magnetic duality (S-duality) maps 't Hooft line operators into Wilson line operators, and this gives the usual statements about categorical geometric Langlands. The goal of the present section is to show how these operators can be thought of not just as functors, but as honest operators acting on the Hilbert space $\mathcal{H}$.

The usual picture of line operators in a general two-dimensional TFT is given in Fig. 3. Interpreting line operators as functors, we run an operator T parallel to one of the boundary conditions, say $\mathcal{B}$ of the TFT, as in Fig. 3a. Moving it to the left boundary it defines a new boundary condition $T\mathcal{B}$. On the other hand, moving it to the right boundary it defines a new boundary condition $T^*\mathcal{B}'$, where T^* is the adjoint of T (this appears due to the reversed orientation fo the line operator relative to the boundary condition). Thus, we obtain a functor $\mathrm{Hom}(T\mathcal{B}, \mathcal{B}') \to \mathrm{Hom}(\mathcal{B}, T^*\mathcal{B}')$ induced by T. This is the story that gives the categorical geometric Langlands correspondence. In Fig. 3b, we instead have a *horizontal* line operator. In (c) and (d) of the same figure, we specify additional data $\alpha \in \mathrm{Hom}(\mathcal{B}, T\mathcal{B})$ at the left endpoint of the operator and $\beta \in \mathrm{Hom}(T\mathcal{B}', \mathcal{B}')$ at the right one. The operator on the quantum Hilbert space $\mathcal{H} = \mathrm{Hom}(\mathcal{B}', \mathcal{B})$ associated to T is $\hat{T}$, which is defined as follows. Given $\psi \in \mathcal{H}$, we have $\alpha\psi\beta \in \mathrm{Hom}(T\mathcal{B}', T\mathcal{B}) = \mathrm{Hom}(\mathcal{B}', T^*T\mathcal{B})$, where T^* is the adjoint of T. Since line operators form an algebra, we get a map $w : \mathrm{Hom}(\mathcal{B}', T^*T\mathcal{B}) \to \mathcal{H}$. Define $\hat{T}(\psi) = w\alpha\psi\beta$.

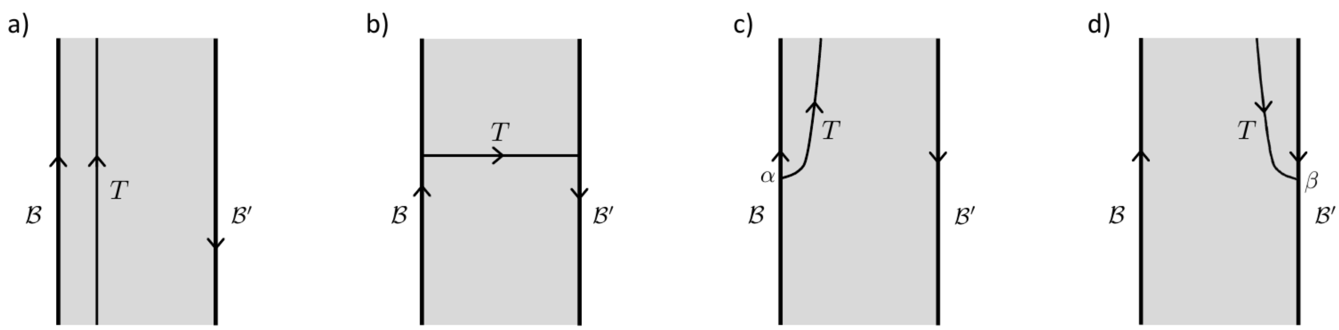

Figure 3: a) A line operator T is parallel to the left boundary of the strip and is oriented compatibly. Moving it to the left boundary, it maps $\mathcal{B}$ to a composite boundary condition $T\mathcal{B}$. This is a line operator when viewed as a functor on the category of branes, as in categorical geometric Langlands. b) In the analytic theory, we take the same line operator T but run it *horizontally* along the strip Σ. Along with some additional data at its endpoints, it can be interpreted as an operator acting on the physical Hilbert space. c), d) The additional data necessary for (b) is an element $\alpha \in \mathrm{Hom}(\mathcal{B}, T\mathcal{B})$ at the left endpoint and an element $\beta \in \mathrm{Hom}(T\mathcal{B}', \mathcal{B}')$ at the right endpoint.

In the analytic Langlands story, we have two extra dimensions beyond the two pictured here. In general, 't Hooft operators or their dual Wilson operators can have support an arbitrary curve γ in $\Sigma \times X$; however, Gaiotto-Witten restrict to those operators with $\gamma = \ell \times p$, where $p \in X$ and ℓ is a curve in Σ. Of course, this is the standard story familiar from geometric Langlands, where the Hecke functors are parametrized by $p \in X$. In addition, we suppose that 't Hooft operators are labeled by a finite-dimensional irreducible representation R of ${}^LG$ (or, equivalently, of ${}^LG_{\mathbb{C}}$). We can denote such an operator as $T_{R,p}$; its dual Wilson operator can be denoted $W_{R,p}$. Thus, putting the last two paragraphs together, we have that $\hat{T}_{R,p}$ are the 't Hooft operators which act on $\mathcal{H}$ associated to a line operator $T_{R,p}$ with the α, β data supplied at its endpoints.

In [2],[3], Hecke operators have the property that they commute with each other and the quantized Hitchin hamiltonians. This follows in the gauge theoretic description for the same reason as the commutativity of the Hitchin hamiltonians themselves.

Proposition 2. 't Hooft operators commute with each other and with the quantum Hitchin hamiltonians.

Physics Proof. A 't Hooft operator $T_{R,P}$ commutes with a Hitchin hamiltonian $H_{P,\alpha}$ because one can assume that the support of α is disjoint from p, so that we can slide $T_{R,p}$ and $H_{P,\alpha}$ past each other without singularity (recall that an element of $\mathcal{A}$ can be viewed as a point operator sitting on a boundary condition, so it can be viewed as a point operator sitting on $\mathcal{B}$ in Fig. 3b). Similarly, given two 't Hooft operators for distinct $p, p' \in X$, clearly $T_{R,p}$ and $T_{R',p'}$ commute. Finally, we can take the limit $p' \to p$ to find that $T_{R,p}$ and $T_{R',p}$ commute as well, without requiring $R = R'$. □

Dual to 't Hooft operators are Wilson operators. Given a 't Hooft operator $T_{R,p}$ we have the dual operator $W_{R,p}$ which is labeled by the same representation R of ${}^LG_{\mathbb{C}}$ and supported at the same point $p \in X$. Classically, Wilson operators have a simple definition. Given a gauge theory on a four-manifold M and a principal ${}^LG_{\mathbb{C}}$ bundle ${}^LE_{\mathbb{C}} \to M$ with connection A, we can associate the vector bundle ${}^LE_R = {}^LE_{\mathbb{C}} \times_{{}^LG_{\mathbb{C}}} R$. Denoting the induced connection also by A, the Wilson operator is just the holonomy of A on ${}^LE_R$ integrated along some oriented path γ in M. If γ is a loop, then we take the trace of the holonomy about γ, and this gives the usual physical Wilson operator. However, since $T_{R,p}$ is a line, not a loop, the Wilson operator is also not the trace of the holonomy on a loop described above. Instead, this operator can be viewed as a mapping between the fibers over the endpoints of the 't Hooft operator. For definiteness, denote these fibers as $\mathcal{F}_{R,p_1}$ and $\mathcal{F}_{R,p_2}$, where p_i are the two endpoints of the Wilson operator $T_{R,p}$ (i.e. $p_i = a_i \times p$, where $a_i \in \Sigma$). We can then write

$$W_{R,p} : \mathcal{F}_{R,p_1} \to \mathcal{F}_{R,p_2}.$$

If we introduce the dual representation to R, R^*, then we can instead view $W_{R,p}$ as a linear function on a representation:

$$W_{R,p} : \mathcal{F}_{R,p_1} \otimes \mathcal{F}_{R^*,p_2} \to \mathbb{C}. \tag{2}$$

However, what we want is a complex valued function on Higgs bundles (more specifically on connections on these bundles), not one on vector spaces. If we could find two vectors, one in each of the fibers on which $W_{R,p}$ acts, then $W_{R,p}(v \otimes w)$ will be the complex-valued function that we could quantize to obtain an operator on $\mathcal{H}$. However, in order to find natural candidates for v and w, we need to describe how the oper condition arises in the gauge-theoretic story, which we will now do. After the following section which will allow us to define the Wilson operators as operators acting on the physical Hilbert space, we can begin to analyze the eigenvalues of the operators heretofore described, finishing the analytic Langlands story as it appears in the physical context.

4 S-duality Enforces the Oper Condition

4.1 The Duals of the Coisotropic Branes

Recall the A-branes $\mathcal{B}_{\mathrm{cc}}$ and $\overline{\mathcal{B}}_{\mathrm{cc}}$ on $\mathcal{M}_H(X, G)$. Under mirror symmetry, these branes should have duals in the B-model of $\mathcal{M}_H(X, {}^LG)$. This space parametrizes flat ${}^LG$-bundles over X with connection $\mathscr{A} = A + i\varphi$, where (A, φ) are the unitary connection and Higgs field of the A-model bundle in $\mathcal{M}_H(X, G)$. In general, B-branes are elements of $\mathrm{D^bCoh}(\mathcal{M}_H(X, {}^LG))$, complexes of coherent sheaves on $\mathcal{M}_H(X, {}^LG)$. Note that when we say B-model without further qualification, we mean the B-model in complex structure

J of the Hitchin moduli space. Now, the Hitchin moduli space is hyper-Kähler, so it comes equipped with complex structures I, J, K obeying the usual quaternion relations. Moreover, it also comes equipped with Kähler forms $\omega_I, \omega_J, \omega_K$ and complex symplectic forms $\Omega_I = \omega_J + i\omega_K, \ldots$ Recall that in order to define the A-model or the B-model, one needs a symplectic form ω. Thus, we can define three sorts of A- or B-models on $\mathcal{M}_H(X, G)$ (equivalently $\mathcal{M}_H(X, {}^L G)$), each one corresponding to one of the three symplectic forms. A general brane can thus have the structure of an A-brane or a B-brane in each of the three forms; for example, we could have a brane of type (B, B, B), which happens to be the brane supported on a point. The dual of such a brane is a brane of type (B, A, A). In particular, the dual of rank 1 brane supported at a point of the Higgs bundle moduli space is a brane supported on a fiber of the Hitchin fibration with a rank 1 flat CP bundle. These branes are exactly the Hecke eigensheaves of the geometric Langlands program. Now, $\mathcal{B}_{\rm cc}$ and $\overline{\mathcal{B}}_{\rm cc}$ are branes of type (A, B, A). This follows from the Kapustin-Orlov conditions for complex structures I and K; they are B-branes in complex structure J because the curvatures $\pm\frac{1}{2}\omega_J$ of their CP bundles are of type $(1, 1)$ in complex structure J, so these bundles are holomorphic in J. In general, the dual of an (A, B, A)-brane is also an (A, B, A)-brane. A gauge-theoretic explanation involves a duality between two string-theoretic systems.

Taking this for granted, we find that the complex lagrangian submanifold supporting the dual of $\mathcal{B}_{\rm cc}$ parametrizes flat, holomorphic ${}^L G_{\mathbb{C}}$ opers; Gaiotto-Witten denote it $L_{\rm op}$. Similarly, the dual of $\overline{\mathcal{B}}_{\rm cc}$ is supported on $L_{\overline{\rm op}}$, which is a complex lagrangian submanifold that parametrizes flat, antiholomorphic ${}^L G_{\mathbb{C}}$ opers. In fact, we will see that the oper condition naturally arises from S-duality acting on deformed Neumann boundary conditions (to be defined in §4.4) in the four-dimensional gauge theory we are studying. Thus, there is a natural physical explanation for the oper condition that arises from the gauge theory itself. We note that although we defined opers in class for e.g. $SL_2(\mathbb{C})$ or $SL_n(\mathbb{C})$ via an extension of square roots of the canonical bundle of X, there is another, equivalent definition via local conditions on the bundle, which we will now describe. It is these local constraints that are enforced by the S-dual of the deformed Neumann boundary condition, called a deformed Nahm pole.

4.2 Local Definition of Opers

Consider first the case of ${}^L G_{\mathbb{C}} = \mathrm{SL}_2(\mathbb{C})$. The extension condition

$$0 \to K_X^{1/2} \to {}^L E \to K_X^{-1/2} \to 0 \tag{3}$$

which defines the ${}^L G_{\mathbb{C}}$-oper ${}^L E$ implies the existence of $s \in \Gamma(X, {}^L E \otimes K_X^{-1/2})$. Denoting as D the $(1, 0)$ part of the connection, we have an $SL_2(\mathbb{C})$-invariant combination $s \wedge Ds$ which is a global holomorphic function on X. Note that it is nonzero; otherwise, we would have $Ds = as$, and a would define a holomorphic flat connection on $K_X^{-1/2}$, which doesn't exist (if we assume $g > 1$ or introduce a parabolic structure for the case $g \leq 1$). Derivating, we ahve $s \wedge D^2 s = 0$, so s satisfies a second order differential equation

$$D^2 s + ts = 0,$$

where t is some "stress tensor" on X. We can define a set of generators of the algebra of holomorphic functions on the manifold of opers via

$$f_{t,\alpha} = \int_X \alpha t$$

where α is a (0,1)-form with values in T_X (i.e. a section of $\Omega_X^{(0,1)} \otimes T_X$).

Similarly, taking ${}^L G_{\mathbb{C}} = \mathrm{SL}_n(\mathbb{C})$, we have a global section $s \in \Gamma(X, {}^L E \otimes K_X^{(1-n)/2})$, so that $s \wedge Ds \wedge \cdots \wedge D^{n-1}s$ is a global holomorphic function on C which does not vanish. We can normalize s (up to a multiple of an n^{th} root of unity) so that this function is identically 1, we again have a degree n differential equation

$$D^n s + t_2 D^{n-2} s + \cdots + t_n s = 0.$$

Thus, we can define a set of generators of the algebra of holomorphic functions on the oper manifold by

$$f_{t_k,\alpha} = \int_X \alpha t_k,$$

where α is now a $(0,1)$-form with values in T_X^{k-1}, $k \in \{2, \ldots, n\}$.

For general ${}^L G_{\mathbb{C}}$, suppose given an oper bundle ${}^L E$. We can then consider the associated bundles ${}^L E_R$ for any irreducible representation R of ${}^L G_{\mathbb{C}}$. We now bootstrap the theory for $\mathrm{SL}_2(\mathbb{C})$ to this more general case, as follows. By the definition of an oper, the structure group of ${}^L E_R$ as a holomorphic vector bundle reduces to a rank 1 subgroup $H_{\mathbb{C}} \subset {}^L G_{\mathbb{C}}$, and this subgroup is a copy of either $\mathrm{SL}_2(\mathbb{C})$ or $\mathrm{SO}(3, \mathbb{C})$, depending on ${}^L G_{\mathbb{C}}$ and R. Let R_n be the n-dimensional irreducible representation of $H_{\mathbb{C}}$. Thus, $R \cong \bigoplus_{n=0}^{\infty} Q_n \otimes R_n$, where the Q_n are vector spaces, almost all of which vanish. Suppose $N \in \mathbb{N}$ is the largest integer for which Q_n is nonzero. Then, $Q_N \cong \mathbb{C}$ is 1-dimensional, and we can thus write $R \cong R_N \oplus \bigoplus_{n=0}^{N-1} Q_n \otimes R_n$. A highest weight vector of R_N is then a highest weight vector of ${}^L G_{\mathbb{C}}$. Furthermore, the bundle ${}^L E_R$ also has a decomposition as a holomorphic vector bundle

$${}^L E_R = {}^L E_{R_N} \oplus \bigoplus_{n=0}^{N-1} Q_n \otimes {}^L E_{R_n}.$$

We didn't discuss opers for general G in class, so let's make a quick digression on this. We recall that given any simple complex Lie group G, there is a **principal embedding** of Lie algebras $\rho : \mathfrak{su}_2 \to \mathfrak{g}$. This embedding descends to one of Lie groups; however, the corresponding principal subgroup can be either $\mathrm{SL}_2(\mathbb{C})$ or $\mathrm{SO}(3, \mathbb{C})$, as mentioned above. A G-oper is then defined to be a flat, holomorphic G-bundle E that is equivalent to a principal embedding of an $\mathrm{SL}_2(\mathbb{C})$ oper bundle. For example, if $G = \mathrm{SL}_n(\mathbb{C})$, this means that the oper bundle is the $(n-1)^{\text{st}}$ symmetric tensor power of an extension of the form of Equation 3. It therefore has a subbundle isomorphic to $K_X^{(n-1)/2}$, as we saw above:

$$0 \to K_X^{(n-1)/2} \to E \to \cdots . \tag{4}$$

An antiholomorphic G-oper is defined similarly.

For each $n \in \mathbb{N}$, we get from Equation 4 an image s_{R_n} of the vector space Q_n into the space $\Gamma(X, {}^L E_R \otimes K_X^{(1-n)/2})$, or, equivalently, a holomorphic function

$$s_{R_n} : Q_n \otimes K_X^{(n-1)/2} \to {}^L E_R.$$

The "highest weight" object s_{R_N}, which we will henceforth denote just as s_R, is of critical importance. It is possible to define a system of differential equations for in a similar manner as in the cases of $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SL}_n(\mathbb{C})$ for the section s_R, but this is not necessary for the present paper.

4.3 Nahm Poles Impose the Local Constraints

A **Nahm pole** is by definition a certain solution of the Nahm equations, which we will now describe. Consider gauge theory on an oriented riemannian four-manifold M which in addition to the gauge field A has an adjoint-valued one-form ϕ. Specializing to $M = \mathbb{R}^3 \times \mathbb{R}_+$, where $\mathbb{R}_+$ is the half-line $y \geq 0$. Denote by $\vec{\phi}$ the part of ϕ normal to the boundary at $y = 0$. Nahm's equations are then

$$\frac{\vec{\phi}}{\mathrm{d}y} + \vec{\phi} \times \vec{\phi} = 0.$$

These equations have a singular solution, which is called the **Nahm pole**. Pick a principal embedding $\rho : \mathfrak{su}_2 \to \mathfrak{g}$, given by a triple of elements $\vec{t} \in \mathfrak{g}$ that satisfy $[t_i, t_j] = \varepsilon_{ijk} t_k$. The solution is then

$$\vec{\phi} = \frac{\vec{t}}{y}.$$

Now, suppose we are in the setting of the present paper, and consider again the moduli space $\mathcal{M}_H(X, G)$. Further, suppose that the underlying spacetime manifold is $M = \Sigma \times X$ as before, but now we set $\Sigma = \mathbb{R} \times \mathbb{R}_+$, where $\mathbb{R}_+$ has the coordinate y as above and X has complex coordinate z. Skipping over some details which are covered in [6], suppose we (somehow) reduce the theory on $\mathbb{R}$ to a three-dimensional theory. Let the complex flat connection on a G-bundle on M be given by $\mathscr{A}$, which explicitly has the form

$$\begin{aligned} \mathscr{A}_z &= \frac{t_+}{y} \\ \mathscr{A}_{\bar{z}} &= 0 \\ \mathscr{A}_y &= \frac{t_3}{y}, \end{aligned}$$

where $t_+ = t_1 + i t_2$ and $\vec{t}$ describes the the principal $\mathfrak{su}_2$ embedding into $\mathfrak{g}$. Suppose now that $\mathfrak{g} = \mathfrak{su}_2$. Since the connection is flat, it can be described by a formula $\mathrm{d} + \mathscr{A} = g \,\mathrm{d}\, g^{-1}$, where $g \in G$. Explicitly, we can take

$$g = \begin{bmatrix} y^{-1/2} & -z y^{-1/2} \\ 0 & y^{1/2} \end{bmatrix},$$

which gives

$$\mathscr{A}_z = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{1}{y} \tag{5}$$

$$\mathscr{A}_{\bar{z}} = 0 \tag{6}$$

$$\mathscr{A}_y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{2y}. \tag{7}$$

Now, what is the condition that must be placed on a complex flat bundle E so that it can be placed in the form given for $\mathscr{A}$ above, near $y = 0$? Restrict E to $X \cong X \times y$ for fixed $y > 0$ (also denoted E), and consider solutions s of the equation $\mathcal{D}_y s = 0$ that vanish like $y^{1/2}$ as $s \to 0$, where $\mathcal{D}_y$ is just the covariant derivative in the y-direction. These solutions span a rank one subbundle $L \subset E$, and any such s is a multiple of

$$y^{1/2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

If we regard E as a flat bundle over X, then L is a holomorphic subbundle, since the object just defined is a holomorphic section of L: It is annihilated by the covariant derivative along $\bar{z}$ (and its associated modification similar to the one given for D/Dz in the next sentence; see [6] §3.1 for details). However, it is not annihilated by $D = D/Dz - [\varphi, \cdot]/2\zeta$; rather, we have

$$s \wedge Ds = 1,$$

which is exactly the differential equation associated to the local constraints for an $\mathrm{SL}_2(\mathbb{C})$-oper. The meaning of the expression for D requires explanation. After introducing a complex coordinate z on X, we can write $\phi_2\,\mathrm{d}x^2 + \phi_3\,\mathrm{d}x^3 = \varphi\,\mathrm{d}z + \bar{\varphi}\,\mathrm{d}\bar{z}$, where we suppose the real coordinates on X are x^2 and x^3. ζ is a parameter which parametrizes the $\mathbb{CP}^1$ of complex structures on $\mathcal{M}_H(X, G)$. There is a similar story for $G = \mathrm{SU}_n$ or $G = \mathrm{SL}_n(\mathbb{C})$, that will require more detail from [6] than we would like to include. The conclusion is that the oper conditions discussion in the previous section are likewise recovered by Nahm pole boundary conditions.

Now, suppose we are given an $\mathrm{SL}_2(\mathbb{C})$-oper on X satisfying the local conditions of §4.2. We will now show that we can construct a solution of the Nahm pole boundary conditions. As before, given this oper E, we have a holomorphic subbundle $L \subset E$. We can pass to a gauge in which $\mathscr{A}_{\bar{z}}$ is spanned by vectors whose upper component vanishes, as above, so that $\mathscr{A}_{\bar{z}}$ is lower triangular. The condition $s \wedge Ds \neq 0$ implies that the upper right matrix element of $\mathscr{A}_z$ is nonzero, and we can gauge transform again to set it equal to 1. Pull back E from X to $X \times \mathbb{R}_+$, so we have a flat connection on $\mathbb{R}_+$ that does not depend on y. Finally, make the gauge transformation

$$g = \begin{bmatrix} y^{-1/2} & 0 \\ 0 & y^{1/2} \end{bmatrix}.$$

After this, $\mathscr{A}_z$ has the singular behavior of the solution given in Equation 5, while $\mathscr{A}_{\bar{z}}$ is nonsingular since it's lower triangular. This gauge transformation gives $\mathscr{A}_y$ precisely as in the solution of Equation 5. Thus, $\mathrm{SL}_2(\mathbb{C})$-opers correspond to the solutions of the Nahm pole boundary conditions in three dimensions modulo less singular terms.

4.4 Deformed Neumann Boundary Conditions

Returning to the four-dimensional twisted SYM we've been studying, we assert that $\mathcal{B}_{\mathrm{cc}}$ can be derived from a **deformed Neumann boundary condition**. We recall that ordinary Neumann boundary conditions for a gauge field A with curvature F require $n^i F_{ij} = 0$, where n is the normal vector to the boundary. Deformed Neumann boundary conditions express $n^i F_{ij}$ in terms of boundary values of other fields.

At generic Ψ, the boundary theory associated to such a boundary condition is a Chern-Simons theory with a complex connection $\mathscr{A}$ with curvature $\mathscr{F} = \mathrm{d}\mathscr{A} + \mathscr{A} \wedge \mathscr{A}$ and action

$$I = \frac{\Psi}{4\pi}\int_N \left(\mathrm{tr}(\mathscr{A} \wedge \mathrm{d}A + \frac{2}{3}\mathscr{A}^3)\right),$$

where N is the boundary or a portion of the boundary of the spacetime manifold M. We remark that along ∂N, we can consider a junction between the deformed Neumann boundary condition leads to I and some other boundary condition. For suitable choices of this second boundary condition, a CFT current algebra will appear along ∂N with level $\Psi - h^\vee$, where $h^\vee$ is the dual Coxeter number of G. Furthermore, the $-h^\vee$ is a 1-loop correction to the theory, exactly as it appears in the renormalization of the level k in two-dimensional CFT.

Now, there is a procedure to take $\Psi \to 0$ which gives a holomorphic-topological boundary condition, in the sense that the boundary condition will vary holomorphically along X and topologically along a one-manifold S, with $N = S \times C$. The procedure is to take the limit $\Psi \to 0$ keeping fixed $\varphi = \frac{\Psi}{4\pi}\mathscr{A}_z \, \mathrm{d}z$.[3] Under this procedure, the action I transforms to

$$I_{\varphi\mathscr{F}} = \int_N \mathrm{tr}(\varphi_z \mathscr{F}_{t\bar{z}}) \, \mathrm{d}t \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z},$$

where φ is the momentum conjugate to $\mathscr{A}_{\bar{z}}$, i.e. it is given by

$$\varphi = \frac{\partial \mathcal{L}}{\partial(\partial_t \mathscr{A}_{\bar{z}})},$$

where $\mathcal{L}$ is the lagrangian of I. The calculation of this result isn't very illuminating, so we will omit it. This new action describes a topological gauged quantum mechanics on the cotangent bundle to the space of $(0,1)$-connections on X. In two-dimensional terms, the A-model of $\mathcal{M}_H(X, G)$ with a $\mathcal{B}_{\mathrm{cc}}$ boundary condition is related to analytically continued quantum mechanics on $\mathcal{M}_H(X, G)$. This is a heuristic analysis. The (physics) proof that $\mathcal{B}_{\mathrm{cc}}$ boundary conditions are encoded by deformed Neumann ones occupies about half of §7 of [5]; we will not include this here. Moreover, the gauge-invariant local operators on N are gauge-invariant polynomials $P(\varphi)(z)$ which descend to the Hitchin hamiltonians in the two-dimensional A-model. This phenomenon should further suggest the link to $\mathcal{B}_{\mathrm{cc}}$.

It turns out that S-duality maps deformed Neumann boundary conditions to Nahm pole boundary conditions; however, the details of this are quite complicated and occupy much of §7 of [5]. Unfortunately, it would take too much space to include this here. Nonetheless, we remark that this is exactly the situation we want for analytic Langlands: On the one hand, we start with deformed Neumann boundary conditions which are equivalent to the $\mathcal{B}_{\mathrm{cc}}$ and $\overline{\mathcal{B}}_{\mathrm{cc}}$ conditions along with 't Hooft or Hecke operators and their eigenvalues which act on the physical Hilbert space defined by $\mathcal{B}_{\mathrm{cc}}$ and $\overline{\mathcal{B}}_{\mathrm{cc}}$. Dually, we have the Nahm pole boundary conditions which give rise to ${}^L G$-opers associated to the given Hilbert space along with Wilson operators and their associated eigenvalues. The Wilson operators don't seem to have a natural place in the mathematical story at this time, and it is interesting to ponder what is their and their eigenvalues' mathematical significance for analytic Langlands.

5 Eigenvalues of the Operators

5.1 Eigenvalues of the Hitchin Hamiltonians

We saw in §4.1 that the B-model dual of $\mathcal{B}_{\mathrm{cc}}$ was $\mathcal{B}_{\mathrm{op}}$, whose support was the lagrangian submanifold L_{op}. Similarly, the dual of $\overline{\mathcal{B}}_{\mathrm{cc}}$ is supported on $L_{\overline{\mathrm{op}}}$, which parametrizes antiholomorphic ${}^L G$-opers. Etingof-Frenkel-Kazhdan conjecture in [2],[3] that these two lagrangians have isolated and transverse intersections

[3] Note that we can't just set $\Psi = 0$, since a (contour) path integral with zero action doesn't make sense. Moreover, there is no way to take the limit $\Psi \to 0$ while preserving the topological invariance on N.

(this is known to be true for $\mathrm{SL}_2(\mathbb{C})$). This actually follows from S-duality; however, this follows from a detailed analysis of B-model quantum mechanics which we have chosen not to include. The conclusion is that a necessary condition for the hermitian form on $\mathcal{H}$ to be positive definite is exactly the statement of this conjecture.

Suppose now that the center of ${}^LG$ is trivial, which allows us to analyze the dual theory in a σ-model without considering $Z({}^LG)$ gauge fields. Let $\Upsilon = L_{\mathrm{op}} \cap L_{\overline{\mathrm{op}}}$, so that a point in Υ is a flat ${}^LG_{\mathbb{C}}$-bundle that is both a holomorphic and an antiholomorphic oper. Furthermore, assuming that the intersection points are isolated and transverse, $\mathcal{H} = \mathrm{Hom}(\mathcal{B}_{\overline{\mathrm{op}}}, \mathcal{B}_{\mathrm{op}})$ has a basis with one basis vector ψ_u for each $u \in \Upsilon$.

It has been conjectured in [2],[3] that a bundle which is both a holomorphic and antiholomorphic oper must be real. Indeed, this follows from the physics. Suppose $u \in \Upsilon$ corresponds to a copmlex flat bundle E that is an oper both holomoprhically and antiholomoprhically. Then its complex conjugate $\overline{E}$ is also such an oper. If E and $\overline{E}$ are not gauge-equivalent, then $\overline{E}$ corresponds to a point $\bar{u} \in \Upsilon$ distinct from U. Thus, they will correspond to distinct basis vectors ψu and $\bar{u}$ of $\mathcal{H}$; moreover, these will be exchanged by the symmetry $\Theta\tau$. The natural B-model pairing is diagonal in the basis of the ψ_u, so that we can normalize the basis vectors such that $(\psi_u, \psi_{u'}) = \delta_{uu'}$ for any $u, u' \in \Upsilon$. Thus, if $\Theta\tau$ exchanges two distinct basis vectors, it follows that they are both 0 vectors for the hermitian inner product defined in §3.2. As mentioned above, the inner product is positive definite; hence, it follows that E must be equal to $\overline{E}$ up to a gauge transformation. In particular, E must be real.

Now, we can predict the spectrum of the Hitchin hamiltonians viewed as operators on $\mathcal{H}$. Let $H_{P,\alpha} \in \mathcal{A} = \mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$ be such a quantized Hitchin hamiltonian. Duality identifies $\mathcal{A}$ with $\mathrm{Hom}(\mathcal{B}_{\mathrm{op}}, \mathcal{B}_{\mathrm{op}})$; thus, it identifies $H_{P,\alpha}$ with a holomoprhic function $f_{P,\alpha}$ on L_{op}. Acting on a basis vector ψ_u that corresponds to a point $u \in \Upsilon$, $H_{P,\alpha}$ just acts by multiplication by the corresponding eigenvalue $f_{P,\alpha}(u)$. Similarly, given $H_{\bar{P},\bar{\alpha}} \in \overline{\mathcal{A}} = \mathrm{Hom}(\overline{\mathcal{B}}_{\mathrm{cc}}, \overline{\mathcal{B}}_{\mathrm{cc}})$, we get that it corrresponds to a holomorphic function $f_{\bar{P},\bar{\alpha}}$ on $L_{\overline{\mathrm{op}}}$. It acts on ψ_u as multiplication by $f_{\bar{P},\bar{\alpha}}$.

In order to include the center of ${}^LG$, we need to slightly refine the description of the dual of the $\mathcal{B}_{\mathrm{cc}}$ (resp. $\overline{\mathcal{B}}_{\mathrm{cc}}$) brane. In particular, $\mathcal{B}_{\mathrm{op}}$ is the dual of $\mathcal{B}_{\mathrm{cc}}$ in the σ-model of $\mathcal{M}_H(X, {}^LG)$, but the low energy description of this model also contains a $Z({}^LG)$ gauge field. Recall the local oper condition discussed in §4.2, where we were able to normalize an object s (guaranteed to exist by virtue of the Nahm pole boundary condition) up to an n^{th} root of unity for the case of ${}^LG = \mathrm{SL}_n(\mathbb{C})$. Such a root of unity is an element of $Z({}^LG)$, so what is happening is that the $Z({}^LG)$ gauge invariance is trivialized along the boundary via a particular choice of normalized S. When we quantize the theory on a strip, we actually have two choices of s, s_ℓ and s_r, at the left and right boundaries of the strip, respectively. We can make a global gauge transformation by an element $x \in Z({}^LG)$ which acts on the pair of trivializations by $(s_\ell, s_r) \mapsto (xs_\ell, xs_r)$. We should then consider such pairs equivalent and mod out by this equivalence relation. This is discussed in more detail in [5] and [6]; we just wanted to give some flavor for what was going on.

5.2 Eigenvalues of Wilson and 't Hooft Operators

It was shown in Proposition 2 that the 't Hooft operators commute with the Hitchin hamiltonians; hence, they can be diagonalized in the same basis of states $\psi_{u,\varepsilon}$ for $u \in \Upsilon$ (assuming again that the center of ${}^LG$ is trivial for simplicity; we could generalize to the case that it isn't without too much difficulty). Recall Equation 2 from the end of §3.4.3. We had obtained a linear function on a vector space, while we want a complex-valued function of connections. In order to obtain this, we just need to supply two vectors $v \in \mathscr{F}_{R,p_1}$ and $w \in \mathscr{F}_{R^*,p_2}$, so that $W_{R,p}(v \otimes w)$ will be the complex-valued function that we could then quantize.

Luckily, a natural candidate was supplied in §4.2, namely $s_R : K_X^{(N-1)/2} \to {}^L E_R$. Thus, given a vector $v \in K_{X,p}^{(N-1)/2}$, we can define $s_R(v) \in {}^L E_{R,p}$ for any holomorphic oper with associated bundle ${}^L E_R$. Similarly, given an antiholomorphic oper with associated bundle ${}^L E_{R'}$, we have a function $\bar{s}_{R'} \overline{K}_X^{(N-1)/2} \to {}^L E_{R'}$. Thus, given $w \in K_{X,p}^{(N-1)/2}$, we get $\bar{s}_{R'}(w) \in {}^L E_{R',p}$. In the case of a bundle ${}^L E_R$ over $\Sigma \times X$ that is a holomorphic oper on the left boundary an antiholomorphic one on the right boundary, we can apply these constructions on each of the boundaries to obtain the function

$$W_{R,p,v\otimes w} = W_{R,p}(s_R(v) \otimes \bar{s}_{R^*}(w)).$$

This is finally a complex-valued function on connections that can be quantized to get a quantum operator on physical states, as promised in §3.4.3.

It is actually quite simple to diagonalize this operator (we use the same notation for the quantum operator that corresponds to the classical function $W_{R,p,v\otimes w}$). Flat connections that satisfy the boundary counditions are in one-to-one correspondence with the basis of states ψ_u that diagonalize the Hitchin hamiltonians. The Wilson operators are diagonal in this basis, and the eigenvalue of $W_{R,p,v\otimes w}$ on a given ψ_u is just the dual pairing $(s_R(v), \bar{s}_{R^*}(w))$ between the dual vector spaces ${}^L E_{R,p}$ and ${}^L E_{R^*,p}$, since flat connections on $\Sigma \times X$ are just pullbacks from X.

A dual 't Hooft operator defined using S-duality will have the same eigenvalues as its associated Wilson operator; this is a basic feature of the notion of S-duality. However, the question then arises *which* 't Hooft operator in the A-model corresponds to a given Wilson operator in the B-model. In particular, what are the endpoints of the operator $T_{R,p}$? This is actually a subtle point. We will sketch the story but not go over it in detail (the details are worked out in §4.3 of [5]). The idea is that the dual 't Hooft operators produce a jump in the fields and in the associated Higgs bundle in the sense that the Higgs bundles on different sides of a given 't Hooft operator are not isomorphic. This leads us to the notion of a "Hecke modification" of a Higgs bundle and in general to the collection of all such. Further, this suggests that we can view a 't Hooft operator as a boundary or interface between two copies of the A-model (below and above it). Generalizing slightly, we can also view it as a brane of type (B, A, A) in the A-model of a product $\mathcal{M}_H(X, G) \times \mathcal{M}_H(X, G)$ after "folding" along $T_{R,p}$. It then takes some careful analysis of this brane to find that the quantized operator acting on $\mathcal{H}$, the quantization can be interpreted as or represented by an integral kernel which is a half-density on $\mathcal{M}(X, G) \times \mathcal{M}(X, G)$ (recall that geometric quantization quantizes the cotangent bundle $T^*\mathcal{M}(X, G)$). After a lengthy analysis in §4.3.3 of [5], one finds that there is a particular unique holomorphic section λ_0 of $K_{Z_{R,p}}^{1/2}$, where $Z_{R,p}$ is the support of the brane mentioned above. Furthermore, this section (more accurately, a certain modification thereof) can be described by a function δ in the Hitchin hamiltonians which is determined exactly by the dual Wilson operator corresponding to $T_{R,p}$. Unfortunately, this last paragraph is probably difficult to understand without working through the details, but we felt it was at least helpful to give a taste of what was going on with the Wilson-'t Hooft duality.

We feel that the gauge theory picture has in some sense been told, at least at a basic level, as we have now seen both sides of the analytic Langlands correspondence appear in this setting. How the oper condition arises naturally out of the physics has been explained, and answers to some conjectures from [2],[3] have been obtained.

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